European Journal of Operational Research 170 (2006) 72–90

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Production, Manufacturing and Logistics

A fuzzy stochastic single-period model for cash management

Jing-Shing Yao [a], Miao-Sheng Chen [b], Huei-Fu Lu [c,d,*]

[a] *Department of Mathematics, National Taiwan University, Taipei 106, Taiwan, ROC*
[b] *Graduate Institute of Management, Nan Hua University, Dalin, Chiayi 622, Taiwan, ROC*
[c] *Graduate Institute of Management Sciences, Tamkang University, Taipei 251, Taiwan, ROC*
[d] *Department of International Business, De Lin Institute of Technology, Taipei 236, Taiwan, ROC*

Received 20 March 2003; accepted 18 June 2004
Available online 13 September 2004

Abstract

The major purpose of this paper is to apply a stochastic single-period inventory management approach to analyze optimal cash management policies with fuzzy cash demand based on fuzzy integral method so that total cost is minimized. We will find that, after defuzzification, the cash-raising amounts and the total costs between the fuzzy case and the crisp case are slightly different when the variation of cash demand is small. As a result, we point out that the fuzzy stochastic single-period model is one extension of the crisp models. In any case, one may conclude that a conscientious analysis in fuzzy mathematics like that presented in this paper provides a financial decision maker with a deeper insight into the more real cash management problem.

Keywords: Fuzzy sets; Stochastic single-period model; Cash management; Fuzzy integral; Signed distance method

1. Introduction

In real business environments, most financial managers have to determine how much cash to raise for normal day-to-day disbursement or protecting against unanticipated variations from budgeted cash flows in a business cycle, and furthermore to achieve the objective of minimizing expected total cost. Because various types of uncertainties and imprecision are inherent in the environment of cash management, they are classically modeled using the probability theory and therefore unpredictable cash demand is usually regarded as a random variable (D_{ran}) with a p.d.f. ($f(D)$), where $f(D)$ may be estimated by past statistical

* Corresponding author. Address: 2 Fl. No. 6, Alley 7, Lane 185, Section 3, Shinglung Rd., Taipei 116, Taiwan, ROC. Tel.: +886 2 22397245; fax: +886 2 22395501.
E-mail addresses: hflu.chibi@msa.hinet.net, hflu@dlit.edu.tw (H.-F. Lu).

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2004.06.017

data. In realistic situations, however, such estimation is often biased. For example, if big news or shocks occur in the financial market, cash demand for the next business cycle will show an unexpected fluctuation. Therefore, facing the dilemma of shortage or excess, the financial manager must adjust cash balances in accordance with real cash demand and reduce cash tied up unnecessarily in the system without diminishing profit or increasing risk.

The problem of managing cash balance is similar to that of managing physical inventory. Baumol (1952) first applied the EOQ model of inventory management in establishing a target cash balance. However, the Baumol model oversimplified the problem. Most importantly, it assumes that cash inflows and outflows are relatively stable and predictable, and it does not take into account any seasonal or cyclical trends. In the literature after Baumol, cash flow is usually regarded as a prescribed constant or a stochastic variable with time or raising quantity dependence (Tobin, 1958; Miller and Orr, 1966; Marquis and Witte, 1989). Recent studies relating certain storage systems to cash flow management have attracted much attention. Harrison et al. (1983) modeled the cash fund as a Brownian motion reflected at the origin. Harrison and Taksar (1983) considered impulse control policies: When the cash fund is too large, the controller may choose to convert some of his cash into securities; when the amount of cash decreases below some limit level, securities are reconverted into cash. Browne (1995) considered a firm with an uncontrollable cash flow and the possibility of investing in risky stock. In the study of Milne and Robertson (1996), a firm's cash flow is determined by a diffusion process and faces liquidation if the internal cash balance falls below some threshold value. Asmussen and Taksar (1997) and Asmussen and Perry (1998) provided jump diffusion models motivated by finance and general storage applications. Perry (1997) also extended the model of Harrison et al. (1983) by taking into account holding cost and unsatisfied demand cost functions to consider drift control for a two-sided reflected Brownian motion. Nevertheless, so far as we know, the optimal cash management policy for business using the concept of fuzzy cash demand has not been considered.

The cash management problem discussed in this paper is closely related to the single-period stochastic inventory, or "newsvendor," problem, which is a standard problem in the literature of inventory (Johnson and Montgomery, 1974; Hamidi-Noori and Bell, 1982). In such a problem, the management has to set the inventory at the level in which the value of the cumulative distribution function is equal to the cost/price ratio. Differing from previous studies, this paper attempts to develop a fuzzy model that takes the vague cash demand into account in order to provide a useful starting point for establishing a target cash balance in a fuzzy environment. We apply a stochastic single-period inventory management approach to analyze optimal cash balance with the considerations of fuzzy information and random components for cash demand (i.e. hybrid cash demand) so that total cost is minimized.

The rest of this paper is organized as follows. Section 2 states the preliminaries where we define fuzzy integral in Property 4, and employ the signed distance method similar to Yao and Wu (2000) to formulate the single-period model.

In Section 3, fuzzy integral method is employed to establish our fuzzy stochastic single-period model with regard to the cash management issue. After defuzzification, we can obtain the estimated total cost in the fuzzy sense in Formula 1. In Section 3.4, we estimate the fuzzy total cost by using exponential distribution as an example of Formula 1 shown in Theorems 2 and 3. In Section 4, we compare the result obtained from the fuzzy case in Section 3.4 to that of the crisp case with numerical operations. Finally, some characteristics of this model are discussed in Section 5 and concluding remarks are in Section 6.

2. Preliminaries

In order to apply the signed distance and the fuzzy integral method to formulate our problem, the following definitions are provided with some relevant operations.

Definition 1. A fuzzy set $\widetilde{A}$ defined on $\Re = (-\infty, \infty)$, which has the membership function, $\mu_{\tilde{A}}(x) = \begin{cases} \alpha, & a \leqslant x \leqslant b, \\ 0, & \text{otherwise,} \end{cases}$ is called α-level fuzzy interval and denoted by $\tilde{A} = [a, b; \alpha]$, where $a < b$.

Definition 2. By Pu and Liu (1980), a fuzzy set $\tilde{a}$ is defined on $\Re$, which has the membership function, $\mu_{\tilde{a}}(x) = \begin{cases} 1, & x = a, \\ 0, & x \neq a, \end{cases}$ is called a fuzzy point.

Let F_s be the family of fuzzy sets on $\Re$, for each $\widetilde{C} \in F_s$, we have a α-level set $C(\alpha) = \{x | \mu_{\widetilde{C}}(x) \geqslant \alpha\} = [C_l(\alpha), C_r(\alpha)]$ $(0 \leqslant a \leqslant 1)$. For each $\alpha \in [0, 1]$, $C_l(\alpha)$ and $C_r(\alpha)$ are the left and right end points of α-level set $C(\alpha)$ separately and both of $C_l(\alpha)$, $C_r(\alpha)$ exit and are continuous over $[0, 1]$.

For $\widetilde{C} \in F_s$, by decomposition theory and Definition 1, we have

$$\widetilde{C} = \bigcup_{0 \leqslant \alpha \leqslant 1} \alpha I_{C(\alpha)} = \bigcup_{0 \leqslant \alpha \leqslant 1} [C_l(\alpha); C_r(\alpha); \alpha], \tag{1}$$

where $I_{C(\alpha)}$ is a characteristic function of $C(\alpha)$.

Similar to Yao and Wu (2000), we consider the signed distance and ranking of $\widetilde{C}(\in F_s)$, we provide Definition 3 as follows.

Definition 3. For a, $0 \in \Re$, we define the signed distance of a measured from the origin 0 by $d_0(a, 0) = a$.

Remark 1. The interpretation of Definition 3 is: If $a > 0$, distance of a from 0 is $d_0(a, 0) = a$, and if $a < 0$, distance of a from 0 is $-d_0(a, 0) = -a$. Thus, $d_0(a, 0) = a$ is called the signed distance of a from origin 0.

The α-level set $\widetilde{C}(\in F_s)$ is denoted by $C(\alpha) = [C_l(\alpha), C_r(\alpha)]$. From Definition 3, the signed distances from left and right end points $C_l(\alpha)$, $C_r(\alpha)$ to origin 0 are defined by $d_0(C_l(\alpha), 0) = C_l(\alpha)$, and $d_0(C_r(\alpha), 0) = C_r(\alpha)$, respectively. Therefore, the signed distance of the closed interval $[C_l(\alpha), C_r(\alpha)]$ from origin 0 can be defined by $d_0([C_l(\alpha), C_r(\alpha)], 0) = \frac{1}{2}[C_l(\alpha) + C_r(\alpha)]$.

For each $\alpha \in [0, 1]$, $[C_l(\alpha), C_r(\alpha)] \leftrightarrow [C_l(\alpha), C_r(\alpha); \alpha]$ is one-to-one onto mapping, so the signed distance of $[C_l(\alpha), C_r(\alpha); \alpha]$ from $\tilde{0}$ is defined by

$$d([C_l(\alpha), C_r(\alpha); \alpha], \tilde{0}) = d_0([C_l(\alpha), C_r(\alpha)], 0) = \frac{1}{2}[C_l(\alpha) + C_r(\alpha)]. \tag{2}$$

Thus, for each $\widetilde{C} \in F_s$ $(0 \leqslant \alpha \leqslant 1)$, Eq. (2) is a function of α and continuous over $[0, 1]$, we can obtain the integral mean value of the signed distance as

$$\int_0^1 d([C_l(\alpha) C_r(\alpha); \alpha], \tilde{0}) \, d\alpha = \frac{1}{2} \int_0^1 (C_l(\alpha) + C_r(\alpha)) \, d\alpha. \tag{3}$$

According to (1) and (3), we have the following definitions of the signed distance and the ranking of fuzzy sets on F_s.

Definition 4. For each $\widetilde{C} \in F_s$, the signed distance of $\widetilde{C}$ from $\tilde{0}$ is defined by

$$d(\widetilde{C}, \tilde{0}) = \frac{1}{2} \int_0^1 (C_l(\alpha) + C_r(\alpha)) \, d\alpha.$$

Definition 5. For $\widetilde{C}$, $\widetilde{D} \in F_s$, define the ranking on F_s by

$$\widetilde{C} \succ \widetilde{D} \quad \text{iff } d(\widetilde{C}, \tilde{0}) > d(\widetilde{D}, \tilde{0}),$$
$$\widetilde{C} \approx \widetilde{D} \quad \text{iff } d(\widetilde{C}, \tilde{0}) = d(\widetilde{D}, \tilde{0}).$$

Using Definition 5 and the order relations $>$, $=$ on $\Re$, we have the following two properties:

Property 1. *For $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C} \in F_s$, the order relations $\succ$, $\approx$ on F_s satisfy the following axioms*: (1) $\widetilde{A} \succ\approx \widetilde{A}$; (2) *if* $\widetilde{A} \succ\approx \widetilde{B}$, $\widetilde{B} \succ\approx \widetilde{A}$, *then* $\widetilde{A} \approx \widetilde{B}$; (3) *if* $\widetilde{A} \succ\approx \widetilde{B}$, $\widetilde{B} \succ\approx C$, *then* $\widetilde{A} \succ\approx \widetilde{C}$.

Property 2. *For $\widetilde{A}$, $\widetilde{B} \in F_s$, the order relations $\succ$, $\approx$ satisfy the law of trichotomy. Namely, one and only one of the three relations of $\widetilde{A} \succ \widetilde{B}$, $\widetilde{A} \approx \widetilde{B}$, $\widetilde{B} \succ \widetilde{A}$ must hold.*

From Properties 1 and 2, we know that the order relations $\succ$, $\approx$ on F_s are linear order.

Definition 6. For $\widetilde{A}$, $\widetilde{B} \in F_s$, define the metric ρ by

$$\rho(\widetilde{A},\widetilde{B}) = |d(\widetilde{A},\tilde{0}) - d(\widetilde{B},\tilde{0})| = \left|\frac{1}{2}\int_0^1 (A_1(\alpha) + A_{\rm r}(\alpha) - B_{\rm l}(\alpha) - B_{\rm r}(\alpha))\,{\rm d}x\right|.$$

Property 3. *For $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C} \in \widetilde{F}_s$, metric ρ satisfies the following three metric axioms*: (1) $\rho(\widetilde{A},\widetilde{B}) = O$ *iff* $\widetilde{A} \approx \widetilde{B}$; (2) $\rho(\widetilde{A},\widetilde{B}) = \rho(\widetilde{B},\widetilde{A})$; (3) $\rho(\widetilde{A},\widetilde{B}) + \rho(\widetilde{B},\widetilde{C}) \geqslant \rho(\widetilde{A},\widetilde{C})$.

Proof. By Definitions 4–6, Property 3 can be proved. □

Definition 7. If $\widetilde{A}(\|P\|)$ (with respect to norm $\|P\|$), $\widetilde{B} \in F_s$ and for each $\varepsilon > 0$, there exist $\delta > 0$, when $\|P\| < \delta$, $\rho(\widetilde{A}(\|P\|),\widetilde{B}) < \varepsilon$, then denoted by $\lim_{\|P\|\to 0}\widetilde{A}(\|P\|) = \widetilde{B}$.

In order to infer the appropriate fuzzy calculus, we refer to the Theorem 3.2 of Goetschel and Voxman (1986) and rearrange it as the following Theorem 1:

Theorem 1. *If the fuzzy function $f : [c,d] \to F$ is continuous with respect to the metric D_1, where*

$$D_1(\{|(a(r),b(r),r)|\ 0 \leqslant r \leqslant 1\},\{|(p(r),q(r),r)|\ 0 \leqslant r \leqslant 1\})$$
$$= \sup(\max\{(|(a(r)-p(r)|,|b(r)-q(r)|)|\ 0 \leqslant r \leqslant 1\}),$$

and if for each $x \in [c,d]$, $f(x)$ has the parametric representation given by $\{|(a(r,x),b(r,x),r)|\ 0 \leqslant r \leqslant 1\}$, then fuzzy integral $\int_c^d f(x)\,{\rm d}x$ exists and belongs to F, that is parameterized by $\left\{\left(\int_c^d a(r,x)\,{\rm d}x, \int_c^d b(r,x)\,{\rm d}x, r\right) 0 \leqslant r \leqslant 1\right\}$, where F is the family of fuzzy numbers.

The calculus of Theorem 1 is to utilize the operational analysis of crisp two-dimensional vector. As to the reasons why it cannot be appropriately treated for fuzzy operation, we will discuss in Section 5.2.

As above-mentioned, the α-level set of a fuzzy set $\widetilde{B}$ on F_s is denoted by $B(\alpha) = [B_{\rm l}(\alpha), B_{\rm r}(\alpha)]$, for each $\alpha \in [0,1]$, let $B_{\rm l}(\alpha)$ and $B_{\rm r}(\alpha)$ be the functions of x denoted by $B_{\rm l}(\alpha) = b_{\rm l}(\alpha,x)$ and $B_{\rm r}(\alpha) = b_{\rm r}(\alpha,x)$ separately (see Section 3.3, for $\widetilde{B} = \widetilde{D}$ and $x = D$). According to decomposition theory, we obtain $\widetilde{B} = \bigcup_{0\leqslant\alpha\leqslant 1}[b_1(\alpha,x), b_{\rm r}(\alpha,x);\alpha]$, which has the different present form from Goetschel and Voxman (1986) but there is the same mean.

Let F_s^* be the family of fuzzy set $\widetilde{B}$ on F_s. On F_s^*, for any $c < d$ (d can be attained to $+\infty$) and for each $\alpha \in [0,1]$, $\int_c^d B_{\rm l}(\alpha)\,{\rm d}x = \int_c^d b_{\rm l}(\alpha,x)\,{\rm d}x$ and $\int_c^d B_{\rm r}(\alpha)\,{\rm d}x = \int_c^d b_{\rm r}(\alpha,x)\,{\rm d}x$ exist. Obviously, $F_s^* \subset F_s$.

Let $\widetilde{B} = \bigcup_{0\leqslant\alpha\leqslant 1}[b_1(\alpha,x), b_{\rm r}(\alpha,x);\alpha]$ be a fuzzy number and $f(\widetilde{B}) \in F_s^*$, by extension principle, the membership function of $f(\widetilde{B})$ can be denoted by $\mu_{f(\widetilde{B})}(z) = \sup_{x\in f^{-1}(z)}\mu_{\widetilde{B}}(x)$ and its α-level set is denoted by $f(\widetilde{B})(\alpha) = [f(\widetilde{B})_1(\alpha), f(\widetilde{B})_{\rm r}(\alpha)]$, for each $\alpha \in [0,1]$, let both $f(\widetilde{B})_1(\alpha)$ and $f(\widetilde{B})_{\rm r}(\alpha)$ be the functions of x denoted by $f(\widetilde{B})_1(\alpha) = a(\alpha,x)$ and $f(\widetilde{B})_{\rm r}(\alpha) = b(\alpha,x)$, respectively. From the property of F_s^*, for any $c < d$ (d can be attained to $+\infty$) and for each $\alpha \in [0,1]$, $\int_c^d f(\widetilde{B})_1(\alpha)\,{\rm d}x = \int_c^d a(\alpha,x)\,{\rm d}x$ and $\int_c^d f(\widetilde{B})_{\rm r}(\alpha)\,{\rm d}x = \int_c^d b(\alpha,x)\,{\rm d}x$ exist. Since $F_S^* \subset F_s$, according to the property of F_s, we know that for each $x \in [c,d]$,

$a(\alpha,x)$ and $b(\alpha,x)$ exist and continuous with respect to $\alpha\in[0,1]$. By decomposition theory, we obtain $f(\widetilde{B})=\bigcup_{0\leqslant\alpha\leqslant 1}[a(\alpha,x),b(\alpha,x);\alpha]$, and for each $\xi\in[c,d]$, the fuzzy set $\tilde{f}(\widetilde{B})(\xi)$ defined as $\tilde{f}(\widetilde{B})(\xi)=\bigcup_{0\leqslant\alpha\leqslant 1}[a(\alpha,\xi),b(\alpha,\xi);\alpha]$.

Let $P=\{x_0,x_1,\ldots,x_n\}$ be a partition of $[c,d]$ and $\Delta_i=x_i-x_{i-1}>0$, $i=1,2,\ldots,n$. For each $\xi_i\in[x_{i-1},x_i]$, $i=1,2,\ldots,n$, we can obtain the fuzzy sets as $\tilde{f}(\widetilde{B})(\xi_i)=\bigcup_{0\leqslant\alpha\leqslant 1}[a(\alpha,\ \xi_i),\ b(\alpha,\ \xi_i);\ \alpha]$, $i=1,2,\ldots,n$. Afterward, for each $i\in\{1,2,\ldots,n\}$, $a(\alpha,\ \xi_i)$ and $b(\alpha,\ \xi_i)$ are continuous with respect to $\alpha\in[0,1]$, all of the α-level sets $[a(\alpha,\ \xi_i),b(\alpha,\ \xi_i)]$ of $\tilde{f}(\widetilde{B})(\xi_i)$ exist and $\tilde{f}(\widetilde{B})\xi_i\in F_s$, $i=1,2,\ldots,n$. Since $\Delta_i>0$ and let the fuzzy operator "(+)" represented by "+", we obtain

$$\sum_{i=1}^{n}\Delta_i\tilde{f}(\widetilde{B})(\xi_i)=\bigcup_{0\leqslant\alpha\leqslant 1}\left[\sum_{i=1}^{n}a(\alpha,\xi_i)\Delta_i,\sum_{i=1}^{n}b(\alpha,\xi_i)\Delta_i;\alpha\right]\in F_s,\quad i=1,2,\ldots,n.$$

Let $\widetilde{F}_1=\bigcup_{0\leqslant\alpha\leqslant 1}[\int_d^c a(\alpha,x)\,\mathrm{d}x,\int_c^d b(\alpha,x)\,\mathrm{d}x;\alpha]$ and $\widetilde{F}_I\in F_s$, because $f(\widetilde{B})\in F_s^*$, it indicates that for each $\alpha\in[0,1]$, both $\int_c^d a(\alpha,x)\,\mathrm{d}x$ and $\int_c^d b(\alpha,x)\,\mathrm{d}x$ exist.

According to the definition of crisp definite integral, we have $\int_c^d a(\alpha,x)\,\mathrm{d}x=\lim_{\|P\|\to 0}\sum_{i=1}^{n}a(\alpha,\xi_i)\Delta_i$, and $\int_c^d b(\alpha,x)\,\mathrm{d}x=\lim_{\|P\|\to 0}\sum_{i=1}^{n}b(\alpha,\xi_i)\Delta_i$, where norm $\|P\|=\max_{1\leqslant i\leqslant n}\Delta_i$. For each $\alpha\in[0,1]$, let $P=\{x_0,x_1,\ldots,x_n\}$, provide the positive $\Delta_i=x_i-x_{i-1}$, $i=1,2,\ldots,n$, and for each $\varepsilon>0$, there exist $\delta_j(\alpha)>0$, $j=1,2$, such that the following equations hold:

$$\|P\|<\delta_1(\alpha),\left|\sum_{i=1}^{n}a(\alpha,\xi_i)\Delta_i-\int_c^d a(\alpha,x)\,\mathrm{d}x\right|<\varepsilon;\tag{4}$$

$$\|P\|<\delta_2(\alpha),\left|\sum_{i=1}^{n}b(\alpha,\xi_i)\Delta_i-\int_c^d b(\alpha,x)\,\mathrm{d}x\right|<\varepsilon.\tag{5}$$

Let $\delta_j=\inf_{0\leqslant\alpha\leqslant 1}\delta(\alpha)$, $j=1,2$. If $\delta_j=0$ then $0<\max_{1\leqslant i\leqslant n}\Delta_i=\|P\|\leqslant\inf_{0\leqslant\alpha\leqslant 1}\delta_j(\alpha)=0$. At the time only $\max_{1\leqslant i\leqslant n}\Delta_i=0$ satisfies the above condition. Namely, $\Delta_i=0$ for $i=1,2,\ldots,n$, but that is nonsensical. Thus, $\delta_j>0$, $j=1,2$, must hold. Now let $\delta=\min(\delta_1,\delta_2)>0$, then we know that for each $\alpha\in[0,1]$ and for each $\varepsilon>0$, there exist $\delta>0$, such that if $P=\{x_0,x_1,\ldots,x_n\}$ with $\|P\|<\delta$, then (4) and (5) hold.

Therefore, when $\|P\|<\delta$ and by Definition 6,

$$\begin{aligned}\rho\left(\sum_{i=1}^{n}\Delta_i\tilde{f}(\widetilde{B})(\xi_i),\widetilde{F}_I\right)&=\left|\frac{1}{2}\int_0^1\left[\sum_{i=1}^{n}a(\alpha,\xi_i)\Delta_i+\sum_{i=1}^{n}b(\alpha,\xi_i)\Delta_i-\int_c^d a(\alpha,x)\,\mathrm{d}x-\int_c^d b(\alpha,x)\,\mathrm{d}x\right]\mathrm{d}\alpha\right|\\&\leqslant\frac{1}{2}\int_0^1\left|\sum_{i=1}^{n}a(\alpha,\xi_i)\Delta_i-\int_c^d a(\alpha,x)\,\mathrm{d}x\right|\mathrm{d}\alpha\\&\quad+\frac{1}{2}\int_0^1\left|\sum_{i=1}^{n}b(\alpha,\xi_i)\Delta_i-\int_c^d b(\alpha,x)\,\mathrm{d}x\right|\mathrm{d}\alpha<\varepsilon.\end{aligned}$$

Finally, by Definition 7, we have $\lim_{\|P\|\to 0}\sum_{i=1}^{n}\tilde{f}(\widetilde{B})(\xi_i)\Delta_i=\bigcup_{0\leqslant\alpha\leqslant 1}[\int_c^d a(\alpha,x)\,\mathrm{d}x,\int_c^d b(\alpha,x)\,\mathrm{d}x;\alpha]$ and employ that to define the fuzzy integral $\int_c^d f(\widetilde{B})\,\mathrm{d}\widetilde{B}$ as Property 4.

Property 4. *If* $f(\widetilde{B})=\bigcup_{0\leqslant\alpha\leqslant 1}[a(\alpha,x),b(\alpha,x);\alpha]\in F_s^*$, *where* $f(\widetilde{B})_1(\alpha)=a(\alpha,x)$, $f(\widetilde{B})_{\mathrm{r}}(\alpha)=b(\alpha,x)$, *then* $\int_c^d f(\widetilde{B})\,\mathrm{d}\widetilde{B}=\bigcup_{0\leqslant\alpha\leqslant 1}\left[\int_c^d a(\alpha,x)\,\mathrm{d}x,\int_c^d b(\alpha,x)\,\mathrm{d}x;\alpha\right]$.

[*Note 1*. The reason we adopt the metric ρ of Definition 6 is alluded to Section 5.2.]

3. Constructing a fuzzy stochastic single-period cash management model

3.1. Crisp case of stochastic single-period cash management model

Before developing the fuzzy stochastic single-period model, we first briefly describe the crisp case proposed by Johnson and Montgomery (1974). We then apply the model to determine the optimal cash-raising level, say R^*. The model describes the cash-raising process, in which the financial manager has to decide how much cash should be raised for a single period under uncertain demand and when the objective is to minimize expected total cost. Now the expected total cost in the crisp model can be expressed as the sum of the raising cost, the expected holding cost and the expected shortage or penalty cost that is given by

$$\begin{aligned} E(R) &= C(R-I) + H\int_0^R (R-D)f(D)\,\mathrm{d}D + V\int_R^\infty (D-R)f(D)\,\mathrm{d}D \\ &= a + HR\int_0^R f(D)\,\mathrm{d}D - H\int_0^R g(D)\,\mathrm{d}D + V\int_R^\infty g(D)\,\mathrm{d}D - VR\int_R^\infty f(D)\,\mathrm{d}D, \end{aligned} \tag{6}$$

where $a = C(R-I)$ and $g(D) = Df(D)$, C = raising cost per unit cash balance, I = amount of net cash inflow on hand before the raising decision at the start of the business cycle, R = amount of cash raised by selling marketable securities or by borrowing, which is a decision variable, D = amount of cash demand for disbursement or transaction during a business cycle,

$$D_{\text{ran}} \text{ is a random variable with p.d.f. } f(D), \qquad (※)$$

H = holding cost per unit cash balance at the end of the cycle (i.e. opportunity cost); V = penalty cost or cost of avoiding a shortage per unit cash balance ($V \geqslant C \geqslant 0$).

Once the optimal cash-raising level (R^*) is determined, the amount of cash balance at the start of business cycle will be computed by $\text{Max}(R^* - I, 0)$.

3.2. Hybrid data of cash demand during a business cycle

In this section, we propose the legitimation of using "hybrid data" to describe the uncertain cash demand. Kaufmann and Gupta (1991) indicated that the components of hybrid data are not homogeneous but are a mixture of random components and fuzzy information. When Saade (1994) applied their concept to consider a fuzzy hypotheses testing problem with hybrid data, he postulated two hypotheses: null hypothesis $H_1 : r = \tilde{A} + n$, and alternative hypothesis. $H_2 : r = \tilde{B} + n$, where $\tilde{A}$, $\tilde{B}$ are fuzzy sets and n is a random component. He then tested them according to the value of the observable r. To give an example, when a tax authority wants to survey the yearly average operating income for nationwide commercial firms, they usually use a random sampling method to obtain the sample data. The collection of the statistical data is based on the past accounting information provided by the firms. However, real operating incomes of firms change frequently and such statistical data should be an estimated value. In other words, the real operating income should be in the vicinity of the estimated value that it is provided with the fuzzy characteristics. Meanwhile, such data would possess random characteristics since it is a consequence of random sampling. Namely, the statistical data provided with both characteristics of fuzzy and random are so-called hybrid data.

As the notation ((※)) of crisp case mentioned in Section 3.1, D is the real amount of cash demand during a business cycle and D_{ran} is a random variable with p.d.f. $f(D)$, the statistical data of random variable D_{ran} means the amount of cash demand (D) during a business cycle, and D_{ran} will be set beforehand as D at the beginning of new business cycle for disbursement or transaction. However, the cash demand in a whole business cycle usually varies with the uncertainty of the financial environment. The real amount of cash demand is not necessarily equal to D at the end of business cycle but may vary during the interval $[D-\Delta_1, D+\Delta_2]$, where $0<\Delta_1<D$, $\Delta_2>0$, and Δ_1, Δ_2 may be appropriately determined by the financial manager. Therefore, from the corresponding interval $[D-\Delta_1, D+\Delta_2]$, we can consider the following triangular fuzzy number:

$$\widetilde{D}=(D-\Delta_1, D, D+\Delta_2). \tag{7}$$

And the membership function of $\widetilde{D}$ is,

$$\mu_{\widetilde{D}}(x)=\begin{cases}\frac{x-D+\Delta_1}{\Delta_1}, & D-\Delta_1\leqslant x\leqslant D,\\ \frac{D+\Delta_2-x}{\Delta_2}, & D\leqslant x\leqslant D+\Delta_2,\\ 0, & \text{otherwise},\end{cases} \tag{8}$$

where $0<\Delta_1<D$, $0<\Delta_2$.

Furthermore, considering the random variable D_{ran} in notation ((※)), the amount of cash demand during a business cycle could be regarded as *hybrid data*, which is the mixed data of random components and fuzzy information.

3.3. Estimate total cost based on fuzzy integral with fuzzy probability distribution $f(\widetilde{D})$ and fuzzy cash demand $\widetilde{D}$

In Section 3.2, since the amount of cash demand during a business cycle may be regarded as hybrid data with random components and fuzzy information, according to the characteristic of fuzzy information of hybrid data and by (7), D is fuzzified as a triangular fuzzy number $\widetilde{D}=(D-\Delta_1, D, D+\Delta_2)$, where $0<\Delta_1<D$, $0<\Delta_2$, and the variations (i.e. Δ_1 and Δ_2) can be appropriately determined by the financial manager. The membership function of $\widetilde{D}$ is therefore denoted by (8). On the other hand, according to the characteristic of random components of hybrid data, we can use $\widetilde{D}$ to fuzzify the cash demand D of p.d.f. $f(D)$ and therefore obtain a fuzzy set $f(\widetilde{D})$. Then, using signed distance to defuzzify the fuzzy set $f(\widetilde{D})$, we can obtain the p.d.f. of D_{ran} in the fuzzy sense (see Remark 4). We employ the extension principle to find its membership function $\mu_{f(\widetilde{D})}(z)$. We also fuzzify $g(D)$ as $g(\widetilde{D})=\widetilde{D}(\cdot)f(\widetilde{D})$. Thus from (6), the fuzzy total cost for a fuzzy stochastic single-period model is given by

$$\widetilde{E}(R)=\tilde{a}(+)\left(HR\int_0^R f(\widetilde{D})\,\mathrm{d}\widetilde{D}\right)(-)\left(H\int_0^R g(\widetilde{D})\,\mathrm{d}\widetilde{D}\right)(+)\left(V\int_R^\infty g(\widetilde{D})\,\mathrm{d}\widetilde{D}\right)(-)\left(VR\int_R^\infty f(\widetilde{D})\,\mathrm{d}\widetilde{D}\right), \tag{9}$$

where $\int_0^R f(\widetilde{D})\,\mathrm{d}\widetilde{D}$, $\int_0^R g(\widetilde{D})\,\mathrm{d}\widetilde{D}$, $\int_R^\infty g(\widetilde{D})\,\mathrm{d}\widetilde{D}$, and $\int_R^\infty d(\widetilde{D})\,\mathrm{d}\widetilde{D}$ are the fuzzy integrals in Property 4.

The α-cut of $\widetilde{D}$ is denoted by $D(\alpha)=[D_1(\alpha), D_r(\alpha)]$, $0\leqslant\alpha\leqslant 1$, where $D_l(\alpha)=D-(1-\alpha)\Delta_1(>0)$ and $D_r(\alpha)=D+(1-\alpha)\Delta_2(>0)$ are the left and right end points of α-cut respectively. By the operation of two crisp intervals proposed by Kaufmann and Gupta (1991):

If $a<b$ and $p<q$, then

$$[a,b]+[p,q]=[a+p,b+q];\quad [a,b]-[p,q]=[a-q,b-p];\quad k[a,b]=\begin{cases}[ka,kb], & \text{if } k\geqslant 0,\\ [kb,ka], & \text{if } k\leqslant 0.\end{cases}$$

$[a,a]$ represents a point "a". If $0\leqslant a<b$ and $0\leqslant p<g$, then

$$[a,b]\bullet[p,q]=[ap,bq]. \tag{10}$$

The α-cut of $f(\widetilde{D})$ and $g(\widetilde{D})$ can be expressed as

$$f(\widetilde{D})(\alpha)=\{z|\mu_{f(\widetilde{D})}(z)\geqslant\alpha\}=[f(\widetilde{D})_1(\alpha), f(\widetilde{D})_r(\alpha)],$$

and

$$g(\widetilde{D})(\alpha) = [g(\widetilde{D})_{\rm l}(\alpha), g(\widetilde{D})_{\rm r}(\alpha)] = [D_{\rm l}(\alpha), D_{\rm r}(\alpha)] \bullet [f(\widetilde{D})_{\rm l}(\alpha), f(\widetilde{D})_{\rm r}(\alpha)].$$

From Property 4, we have the α-cut of fuzzy integrals $\int_0^R f(\widetilde{D})\,{\rm d}\widetilde{D}$, $\int_0^R g(\widetilde{D})\,{\rm d}\widetilde{D}$, $\int_R^\infty f(\widetilde{D})\,{\rm d}\widetilde{D}$, and $\int_R^\infty g(\widetilde{D})\,{\rm d}\widetilde{D}$ as follows:

$$\left(\int_0^R f(\widetilde{D})\,{\rm d}\widetilde{D}\right)(\alpha) = \left[\int_0^R f(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D, \int_0^R f(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D\right],$$
$$\left(\int_0^R g(\widetilde{D})\,{\rm d}\widetilde{D}\right)(\alpha) = \left[\int_0^R g(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D, \int_0^R g(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D\right],$$
$$\left(\int_R^\infty f(\widetilde{D})\,{\rm d}\widetilde{D}\right)(\alpha) = \left[\int_R^\infty f(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D, \int_R^\infty f(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D\right],$$
$$\left(\int_R^\infty g(\widetilde{D})\,{\rm d}\widetilde{D}\right)(\alpha) = \left[\int_R^\infty g(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D, \int_R^\infty g(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D\right], \quad \text{where } 0 \leqslant \alpha \leqslant 1.$$

By decomposition theory, fuzzy total cost (9) becomes

$$\widetilde{E}(R) = \bigcup_{0\leqslant\alpha\leqslant1} [\widetilde{E}(R)_{\rm l}(\alpha), \widetilde{E}(R)_{\rm r}(\alpha); \alpha], \tag{11}$$

where

$$\widetilde{E}(R)_{\rm l}(\alpha) = C(R-I) + HR\int_0^R f(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D - H\int_0^R g(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D + V\int_R^\infty g(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D - VR \times \int_R^\infty f(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D, \tag{12}$$

and

$$\widetilde{E}(R)_{\rm r}(\alpha) = C(R-I) + HR\int_0^R f(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D - H\int_0^R g(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D + V\int_R^\infty g(\widetilde{D})_{\rm r}(\alpha)\,{\rm d}D - VR \times \int_R^\infty f(\widetilde{D})_{\rm l}(\alpha)\,{\rm d}D. \tag{13}$$

After using the signed distance method defined in Definition 4 to defuzzify the fuzzy total cost, we obtain the following estimated total cost in the fuzzy sense:

$$\begin{aligned} E^{**}(R) &= d(\tilde{E}(R), \tilde{0}) \\ &= \frac{1}{2}\int_0^1 \left[\widetilde{E}(R)_{\rm l}(\alpha) + \widetilde{E}(R)_{\rm r}(\alpha)\right]{\rm d}\alpha \\ &= C(R-I) + \frac{1}{2}\left[HR\int_0^R \int_0^1 [f(\widetilde{D})_{\rm l}(\alpha) + f(\widetilde{D})_{\rm r}(\alpha)]\,{\rm d}\alpha\,{\rm d}D \right. \\ &\quad - H\int_0^R \int_0^1 [g(\widetilde{D})_{\rm l}(\alpha) + g(\widetilde{D})_{\rm r}(\alpha)]\,{\rm d}\alpha\,{\rm d}D + V\int_R^\infty \int_0^1 [g(\widetilde{D})_{\rm l}(\alpha) + g(\widetilde{D})_{\rm r}(\alpha)]\,{\rm d}\alpha\,{\rm d}D \\ &\quad \left. - VR\int_R^\infty \int_0^1 [f(\widetilde{D})_{\rm l}(\alpha) + f(\widetilde{D})_{\rm r}(\alpha)]\,{\rm d}\alpha\,{\rm d}D\right]. \end{aligned} \tag{14}$$

Formula 1

(a) By fuzzifying cash demand in (6), the fuzzy total cost is (9).
(b) By (10), fuzzy total cost in (9) can be represented as (11).
(c) Using signed distance method to defuzzify the fuzzy total cost in (11), the estimated total cost in the fuzzy sense (14) can be obtained.

3.4. The derivation of main theorems by using exponential distribution as an example of Formula 1

The following is one example of Formula 1. As to the usage of other forms of p.d.f., we can also employ Formula 1 to obtain similar results. Here, we consider that cash demand during a single business cycle follows an exponential distribution, i.e. $f(D)=\frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right)$, $0\leqslant D$, where θ is known and $0<\theta\leqslant 1$. The mean value of $f(D)$ is $\int_0^\infty Df(D)\,\mathrm{d}D=\theta$.

Remark 2. Because θ is the statistical estimation of cash demand from each past business cycle, the measuring unit of cash demand would be consider as \$1000, \$10,000, \$20,000, ... or \$200,000 to enforce the statistical mean value upon the interval $[0,1]$. For example, if the estimations of cash demand from the past five business cycles are \$110,000, \$120,000, \$160,000, \$180,000, and \$190,000, then we divide the measuring unit by \$200,000 and therefore the statistical data becomes $\frac{11}{20},\frac{12}{20},\frac{16}{20},\frac{18}{20}$ and $\frac{19}{20}$, separately, and their mean value is $\frac{1}{5}\left(\frac{11}{20}+\frac{12}{20}+\frac{16}{20}+\frac{18}{20}+\frac{19}{20}\right)=0.76\in[0,1]$. This is regarded as the estimation of θ, with measuring unit, \$200,000. Furthermore, by Fig. 1, which is a legend of exponential distribution, if $D_0=2\theta\ln\frac{1}{\theta}$, then $f(D_0)=\theta$.

[*Note 2*. In this section, the cash demand can be regarded as *hybrid data*, but θ is a fixed value.]

According to Formula 1 and (7), cash demand D is fuzzified as a triangular fuzzy number $\widetilde{D}=(D-\Delta_1,D,D+\Delta_2)$, where Δ_1 and Δ_2 can be appropriately determined by the financial manager, and the membership function of fuzzy cash demand $\widetilde{D}$ is denoted by (8).

Similarly, the α-*level* set of $\widetilde{D}$ is $D(\alpha)=[D_l(\alpha),D_r(\alpha)]$, $0\leqslant a\leqslant 1$, where

$$D_l(\alpha)=D-(1-\alpha)\Delta_1(>0)\quad\text{and}\quad D_r(\alpha)=D+(1-\alpha)\Delta_2(>0). \tag{15}$$

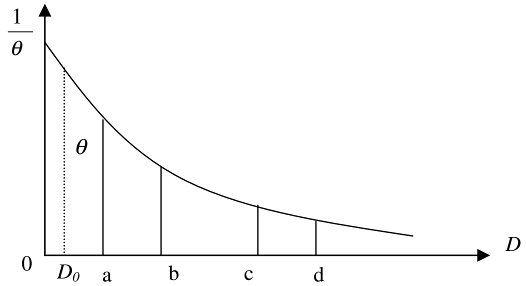

Fig. 1. Probability distribution of $f(D)$.

Remark 3. Formula 1 is the common procedure for finding the estimated total cost in the fuzzy sense under fuzzy p.d.f. $f(\widetilde{D})$. In our cash management problem, on the one hand, cash demand is defined by varying during $[0,\infty)$ so that it is relatively difficult and complex to find out the fuzzy p.d.f. of normal distribution by using Formula 1, and on the other hand, since the exponential distribution shown in Fig. 1 is a concave monotonic decreasing function during $[0,\infty)$, while normal distribution ($N(0,\sigma^2)$) is a convex monotonic decreasing function during $[0,\infty)$, these two distribution functions have analogous shape during $[0,\infty)$. Based on this, it is a better measure to use an exponential distribution as an example of p.d.f. such as in Formula 1 than that of a normal distribution. As regards other forms of p.d.f, we can obtain the analogous results by using Formula 1 as well.

From Formula 1 and Remark 3, we have the following Theorem 2:

Theorem 2 (*Fuzzy case by Formula* 1). *If the p.d.f. of cash demand D follows an exponential distribution* $f(D)=\frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right), 0\leqslant D, 0<\theta\leqslant 1$, *and D is fuzzified as a triangular fuzzy number* $\widetilde{D}$ (*in* (7)), *then the following results can be obtained*:

(a) *The fuzzy total cost is the form in* (9) *with* $f(\widetilde{D})=\frac{1}{\theta}\exp\left(-\frac{1}{\theta}\widetilde{D}\right)$.

(b) *The estimate of total cost in the fuzzy sense is* $E^{**}(R;\theta)=C(R-I)\ \frac{1}{2}HR[F_*(0,R;1)+F_*(0,R;2)]-\frac{1}{2}H[G_*(0,R;1,2)+G_*(0,R;2,1)]+\frac{1}{2}V[G_*(R,\infty;1,2)+G_*(R,\infty;2,1)]-\frac{1}{2}VR[F_*(R,\infty;1)+F_*(R,\infty;2)]$ (*in* (A.14)), *where* $F_*(0,R;j)$, $F_*(R,\infty;j)$; $j=1,2,G^*$ $(0,R;1,2)$, $G_*(0,R;2,1)$, $G_*(R,\infty;1,2)$ *and* $G_*(R,\infty;2,1)$ *are defined by* (A.10)–(A.13) *in the appendix.*

(c) *The optimal solution in the fuzzy sense is*: *if* $B(\theta,\Delta_1,\Delta_2)>0$, $A(\theta,\Delta_1,\Delta_2)>0$, *and* $B(\theta,\Delta_1,\Delta_2)>\theta A(\theta,\Delta_1,\Delta_2)$, *then the optimal cash-raising amount* $R_2^*(\theta;\Delta_1,\Delta_2)=\theta\ln\frac{B(\theta,\Delta_1,\Delta_2)}{\theta A(\theta,\Delta_1,\Delta_2)}$, *where* $A(\theta,\Delta_1,\Delta_2)$ *and* $B(\theta,\Delta_1,\Delta_2)$ *are defined by* (A.18) *in the appendix*, *and the minimum total cost* $E^{**}(R_2^*(\theta;\Delta_1,\Delta_2);\theta)$ *is the result of substituting* $R_2^*(\theta)$ *into R of* $E^{**}(R;\theta)$ *shown in* (*b*).

Proof. See Appendix. □

This means that when the p.d.f. of cash demand is known and the variations of cash demand have been appropriately determined by the financial manager, the optimal cash-raising amount and the minimum total cost formulated under the fuzzy integral method can be found out by Theorem 2.

Theorem 3 (Crisp case). *If the p.d.f. of cash demand D follows an exponential distribution* $f(D)=\frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right)$, $0\leqslant D$, $0<\theta\leqslant 1$, *and total cost is* $E(R)$ (*in* (6)), *then the following results can be obtained*:

(a) *The optimal cash-raising amount is* $R_1^*(\theta)=\theta\ln\frac{H+V}{H+C}$, *where* $0<\theta\leqslant 1$ *and* $V>C$.

(b) *The minimum total cost is* $E(R_1^*(\theta))=(H+C)R_1^*(\theta)+C(\theta-I)$.

Proof. See Appendix. □

$$
\begin{aligned}
f^*(D)\equiv d(f(\widetilde{D}),\tilde{0}) &= \frac{1}{2}\int_0^1\left[\exp\left(-\frac{D+\theta\ln\theta+(1-\alpha)\Delta_2}{\theta}\right)+\exp\left(-\frac{D+\theta\ln\theta-(1-\alpha)\Delta_1}{\theta}\right)\right]\mathrm{d}\alpha \\
&= \frac{\theta}{2\Delta_2}\left[\exp\left(-\frac{D+\theta\ln\theta}{\theta}\right)-\exp\left(-\frac{D+\theta\ln\theta+\Delta_2}{\theta}\right)\right] \\
&\quad -\frac{\theta}{2\Delta_1}\left[\exp\left(-\frac{D+\theta\ln\theta}{\theta}\right)-\exp\left(-\frac{D+\theta\ln\theta-\Delta_1}{\theta}\right)\right],
\end{aligned}
$$

Remark 4. From (A.2) and (A.3) and Definition 4, we have
and

$$\int_0^{\infty} f^*(D)\,\mathrm{d}D = \frac{\theta}{2\Delta_2}\left[1-\exp\left(-\frac{\Delta_2}{\theta}\right)\right] - \frac{\theta}{2\Delta_1}\left[1-\exp\left(-\frac{\Delta_1}{\theta}\right)\right].$$

The financial manager can appropriately determine Δ_1 and Δ_2 to satisfy the following conditions: $D > \Delta_1 > 0$, $\Delta_2 > 0$, and $\frac{\theta}{2\Delta_2}[1-\exp\left(-\frac{\Delta_2}{\theta}\right)] - \frac{\theta}{2\Delta_1}[1-\exp\left(-\frac{\Delta_1}{\theta}\right)] = 1$. Obviously, $f^*(D) > 0$ for all $D > 0$ and $\int_0^{\infty} f^*(D)\,\mathrm{d}D = 1$, then $f^*(D)$ is called the p.d.f. of D_{ran} in the fuzzy sense.

4. Numerical example

The methodology of fuzzification that we have proposed in the preceding sections would be helpful for financial managers to improve their cash-raising decisions and achieve the optimal cash-raising policy for the minimum total cost. In order to specifically illustrate the above procedures, let us consider a hypothetical cash-raising system with the following example.

Example. We illustrate the example by assuming the p.d.f. of cash demand to be $f(D) = \frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right)$, $0 \leqslant D$, $0 < \theta \leqslant 1$. Afterwards, we consider a cash-raising system with the following data: $C = \$0.1$, $I = \$0$, $H = \$0.02$, $V = \$0.2$, and $\$\Delta_1$, $\$\Delta_2$ can be determined by the financial manager, then we employ Theorems 2 and 3 to compute the scenarios of $\theta = 1$, 0.8, 0.6, 0.4, respectively.

Furthermore, in order to find the relative errors in cash-raising amount and total cost between the fuzzy case and crisp case, we let

$$r_{21}(\theta) = \frac{R_2^*(\theta;\Delta_1,\Delta_2) - R_1^*(\theta)}{R_1^*(\theta)} \times 100(\%),$$

$$F_{21}(\theta) = \frac{E^{**}(R_2^*(\theta,\Delta_1,\Delta_2);\theta) - E(R_1^*(\theta))}{E(R_1^*(\theta))} \times 100(\%);$$

these numerical results are presented in Tables 1–4, and furthermore the implications will be separately discussed in the next section.

For simplicity, we use the notation $R_2^*(\theta)$ and $E^{**}(R_2^*(\theta);\theta)$ to respectively replace $R_2^*(\theta;\Delta_1,\Delta_2)$ and $E^{**}(R_2^*(\theta;\Delta_1,\Delta_2);\theta)$ hereafter.

In the illustrated example, all of the parameters such as C, I, H, V, Δ_1 and Δ_2 are measured by millions. The additional explanations for Tables 1–4 are enumerated as follows.

(1) In Table 2, if $\Delta_1 = \Delta_2 = 0.00001$, then the optimal cash-raising amounts and the minimum total costs are almost indifferent between the fuzzy case and the crisp case, (see Section 5.1).

(2) In Tables 2 and 3, for each given $\Delta_2 - \Delta_1 (\geqslant 0)$, when θ increases within interval (0,1], the optimal cash-raising amount, $R_2^*(\theta;\Delta_1,\Delta_2)$, and the minimum total cost, $E^{**}(R_2^*(\theta;\Delta_1,\Delta_2);\theta)$, are decreased.

Table 1
Crisp optimal solutions

θ	1	0.8	0.6	0.4
$R_1^*(\theta)$	0.606136	0.484909	0.363681	0.242454
$E(R_1^*(\theta))$	0.172736	0.138189	0.103642	0.069095

Table 2
Optimal solutions for fuzzy case ($\Delta_1 < \Delta_2$ and Δ_1, Δ_2 in millions)

θ						
	Δ_1	0.00001	0.00001	0.0002	0.003	0.005
	Δ_2	0.00001	0.000012	0.0003	0.004	0.007
	$\Delta_2 - \Delta_1$	0	0.000002	0.0001	0.001	0.002
1	$R_2^*(1)$	0.606136	0.606136	0.60614	0.606183	0.606236
	$E^{**}(R_2^*(1);1)$	0.172736	0.172736	0.172736	0.172734	0.172732
	$r_{21}(1)$ %	0	0.00001	0.0007	0.0078	0.0165
	$F_{21}(1)$ %	0	0	−0.0002	−0.0014	−0.0024
0.8	$R_2^*(0.8)$	0.484909	0.484909	0.484913	0.484957	0.485013
	$E^{**}(R_2^*(0.8);(0.8)$	0.138189	0.138189	0.138189	0.138187	0.138185
	$r_{21}(0.8)$ %	0	0.00002	0.00087	0.0101	0.0215
	$F_{21}(0.8)$ %	0	0	−0.00022	−0.0016	−0.0027
0.6	$R_2^*(0.6)$	0.363681	0.363682	0.363686	0.363733	0.363793
	$E^{**}(R^*(0.6);0.6)$	0.103642	0.103642	0.103641	0.10364	0.103639
	$r_{21}(0.6)$ %	0	0.00002	0.0012	0.0141	0.0306
	$F_{21}(0.6)$ %	0	0	−0.00029	−0.0019	−0.0029
0.4	$R_2^*(0.4)$	0.242454	0.242454	0.242459	0.24251	0.24258
	$E^{**}(R_2^*(0.4);0.4)$	0.069095	0.069095	0.0690-94	0.069093	0.069093
	$r_{21}(0.4)$ %	0	0.00003	0.00175	0.0231	0.0517
	$F_{21}(0.4)$ %	0	0	−0.00043	−0.0021	−0.0021

Table 3
Optimal solutions for fuzzy case ($\Delta_1 < \Delta_2$ and Δ_1, Δ_2 in millions)

θ						
	Δ_1	0.01	0.015	0.02	0.1	0.1
	Δ_2	0.015	0.024	0.03	0.15	0.2
	$\Delta_2 - \Delta_1$	0.005	0.009	0.01	0.05	0.1
1	$R_2^*(1)$	0.606417	0.606686	0.606947	0.61537	0.620167
	$E^{**}(R_2^*(1);1)$	0.172729	0.172728	0.172738	0.173374	0.173529
	$r_{21}(1)$ %	0.0463	0.0908	0.1165	1.5235	2.3149
	$F_{21}(1)$ %	−0.0041	−0.0045	−0. 001	0.3693	0.4588
0.8	$R_2^*(0.8)$	0.485208	0.485503	0.485687	0.495891	0.501306
	$E^{**}(R_2^*(0.8);0.8)$	0.138184	0.138186	0.138199	0.139019	0.139243
	$r_{21}(0.8)$ %	0.0616	0.1225	0.1605	2.2647	3.3816
	$F_{21}(0.8)$ %	−0.0037	−0.0022	0.007	0.6007	0.7624
0.6	$R_2^*(0.6)$	0.364011	0.364348	0.36458	0.377532	0.383921
	$E^{**}(R_2^*(0.6);0.6)$	0.10364	0.103647	0.103665	0.104787	0.105118
	$r_{21}(0.6)$ %	0.0905	0.1833	0.2471	3.8083	5.5653
	$F_{21}(0.6)$ %	−0.0017	0.00486	0.0222	1.1049	1.4243
0.4	$R_2^*(0.4)$	0.242844	0.243266	0.243592	0.261855	0.269971
	$E^{**}(R_2^*(0.4);0.4)$	0.069099	0.069116	0.069144	0.070874	0.071366
	$r_{21}(0.4)$ %	0.1605	0.3348	0.4692	8.0018	11.349
	$F_{21}(0.4)$ %	0.0070	0.0305	0.0715	2.537	3.2877

(3) In Table 3, if $\Delta_1 = 0.1$, $\Delta_2 = 0.2$, $\theta = 0.4$, then $r_{21}(0.4) = 11.349(\%)$ and $F_{21}(0.4) = 3.2877(\%)$. Note that the unit of measurement is millions, so there are significant differences between the fuzzy case and the crisp case.

(4) In Table 4, we may also find the similar features.

Table 4
Optimal solutions for fuzzy case ($\Delta_2 < \Delta_1$, and Δ_1, Δ_2 in millions)

θ	Δ_1	0.000012	0.0003	0.004	0.015	0.15	0.2
	Δ_2	0.00001	0.0002	0.003	0.01	0.1	0.1
	$\Delta_1 - \Delta_2$	0.00001	0.0001	0.001	0.005	0.05	0.1
1	$R_2^*(1)$	0.606136	0.606132	0.6061	0.606	0.611381	0.612328
	$E^{**}(R_2^*(1);1)$	0.172738	0.172744	0.172759	0.172767	0.173701	0.174202
	$r_{21}(1)$ %	−0.00001	−0.00068	−0.0059	−0.022	0.8653	1.0216
	$F_{21}(1)$ %	0	0.000177	0.00212	0.0134	0.5586	0.8487
0.8	$R_2^*(0.8)$	0.484909	0.484905	0.484874	0.484791	0.491999	0.493741
	$E^{**}(R_2^*(0.8);0.8)$	0.138189	0.138189	0.138193	0.138214	0.13936	0.139954
	$r_{21}(0.8)$%	−0.00002	−0.00085	−0.0071	−0.024	1.4623	1.8215
	$F_{21}(0.8)$ %	0	0.00022	0.00277	0.0182	0.0847	1.2769
0.6	$R_2^*(0.6)$	0.363681	0.363677	0.363649	0.363594	0.373851	0.37694
	$E^{**}(R_2^*(0.6);0.6)$	0.103642	0.103642	0.103646	0.10367	0.105156	0.105909
	$r_{21}(0.6)$ %	−0.000023	−0.0011	−0.0088	−0.024	2.7963	3.6455
	$F_{21}(0.6)$ %	0	0.0003	0.00394	0.0276	1.461	2.1874
0.4	$R_2^*(0.4)$	0.242454	0.24245	0.242427	0.242428	0.258758	0.264588
	$E^{**}(R_2^*(0.4);0.4)$	0.069095	0.069095	0.069099	0.06913	0.071296	0.072378
	$r_{21}(0.4)$ %	−0.00003	−0.0017	−0.011	−0.011	6.7244	9.1288
	$F_{21}(0.4)$ %	0	0.00045	0.00668	0.0511	3.1866	4.7523

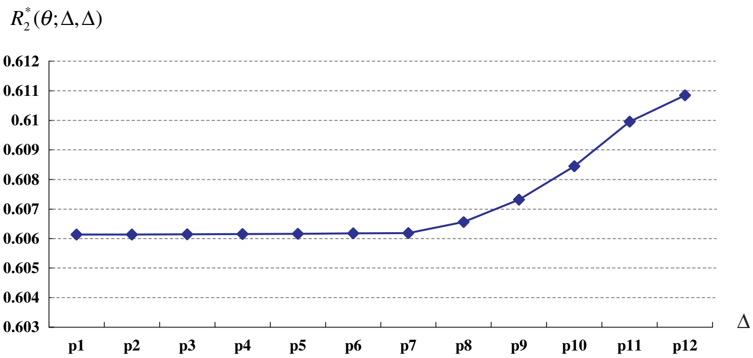

Fig. 2. Graph of $R_2^*(\theta;\Delta,\Delta)$ (scenario: $\theta = 1$, $p1 = 0.0001$, $p2 = 0.001$, $p3 = 0.003$, $p4 = 0.005$, $p5 = 0.007$, $p6 = 0.009$, $p7 = 0.01$, $p8 = 0.03$, $p9 = 0.05$, $p10 = 0.07$, $p11 = 0.09$, $p12 = 0.1$, by Table 1, $R_1^*(1) = 0.606136$).

In addition, when $\Delta \equiv \Delta_1 = \Delta_2$, by rearranging (A.18), we have $A(\theta,\Delta,\Delta) = C + \frac{\theta H\left(\exp\left(\frac{\Delta}{\theta}\right) - \exp\left(-\frac{\Delta}{\theta}\right)\right)}{2\Delta}$ and $B(\theta,\Delta,\Delta) = \frac{\theta}{2}(H+V)\left(\exp\left(\frac{\Delta}{\theta}\right) + \exp\left(-\frac{\Delta}{\theta}\right)\right)$, then Theorem 2(c) becomes $R_2^*(\theta;\Delta_1,\Delta_2) = \theta \ln \frac{B(\theta,\Delta,\Delta)}{\theta A(\theta,\Delta,\Delta)}$, where $0 < \theta \leqslant 1$, and furthermore $E^{**}(R_2^*(\theta;\Delta_1,\Delta_2);\theta)$ (in Theorem 2(c)) can be obtained as well. For $\theta = 1$, 0.6, the varied trends with respect to variation $\Delta, R_2^*(\theta;\Delta_1,\Delta_2)$ and $E^{**}(R_2^*(\theta;\Delta_1,\Delta_2);\theta)$ are shown in Figs. 2–5, respectively.

In Fig. 2, when $\theta = 1$ and the variation Δ from $p1 = 0.0001$ to $p7 = 0.01$, the values of $R_2^*(1;\Delta,\Delta)$ are very close to that of the crisp case ($R_1^*(1) = 0.606136$); and in Fig. 3, there are the same results when $\theta = 0.6$ (see Section 5.1).

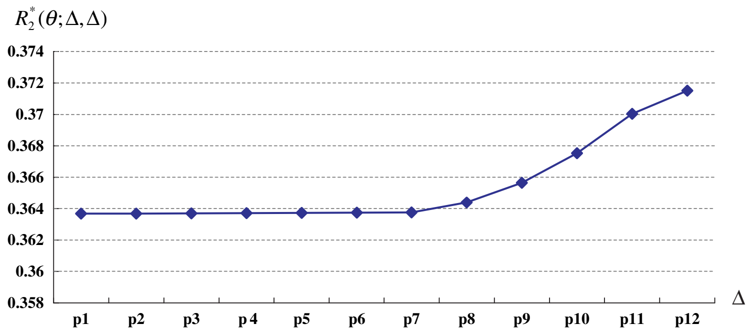

Fig. 3. Graph of $R_2^*(\theta; \Delta, \Delta)$ (scenario: $\theta = 0.6$, $p1 = 0.0001$, $p2 = 0.001$, $p3 = 0.003$, $p4 = 0.005$, $p5 = 0.007$, $p6 = 0.009$, $p7 = 0.01$, $p8 = 0.03$, $p9 = 0.05$, $p10 = 0.07$, $p11 = 0.09$, $p12 = 0.1$, by Table 1, $R_1^*(0.6) = 0.363681$).

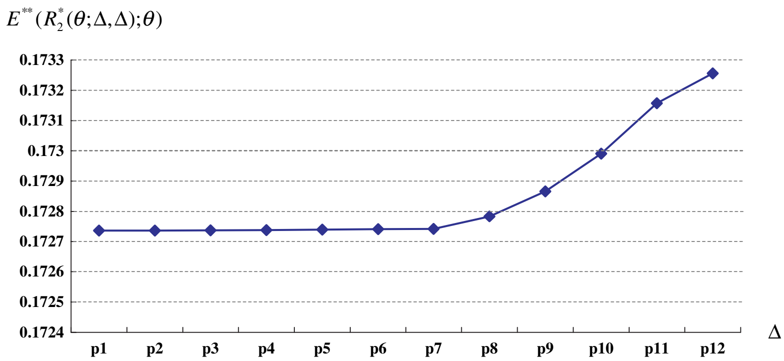

Fig. 4. Graph of $E^{**}(R_2^*(\theta; \Delta, \Delta); \theta)$ (scenario: $\theta = 1$, $p1 = 0.0001$, $p2 = 0.001$, $p3 = 0.003$, $p4 = 0.005$, $p5 = 0.007$, $p6 = 0.009$, $p7 = 0.01$, $p8 = 0.03$, $p9 = 0.05$, $p10 = 0.07$, $p11 = 0.09$, $p12 = 0.1$, by Table 1, $E(R_1^*(1)) = 0.172736$).

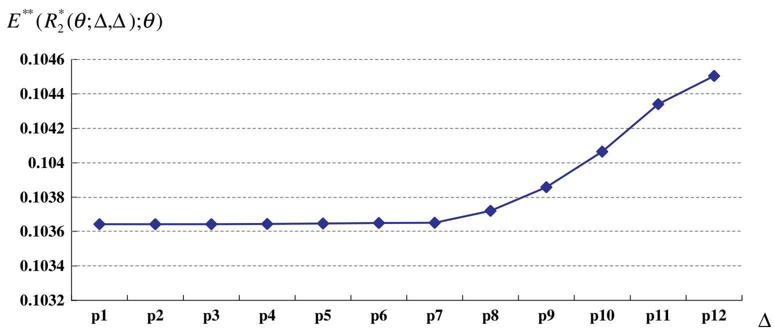

Fig. 5. Graph of $E^{**}(R_2^*(\theta; \Delta, \Delta); \theta)$ (scenario: $\theta = 0.6$, $p1 = 0.001$, $p2 = 0.0001$, $p3 = 0.003$, $p4 = 0.005$, $p5 = 0.007$, $p6 = 0.009$, $p7 = 0.01$, $p8 = 0.03$, $p9 = 0.05$, $p10 = 0.07$, $p11 = 0.09$, $p12 = 0.1$, by Table 1, $E(R_1^*(0.6)) = 0.103642$).

Similarly, in Fig. 4, when $\theta = 1$ and the variation Δ from $p1 = 0.0001$ to $p7 = 0.01$, the values of $E^{**}(R_2^*(1; \Delta, \Delta); \theta)$ are very close to that of the crisp case ($E(R_1^*(1)) = 0.172736$); and in Fig. 5, there are the same results when $\theta = 0.6$ (see Section 5.1).

Our Theorems 2 and 3 constitute the foundations of the fuzzy cash management model we are proposing. The additional discussions are summarized in the next section.

5. Discussions

Let us discuss the features derived from our fuzzy model as below.

5.1. Using triangular fuzzy number $\widetilde{D} = (D - \Delta_1, D, D + \Delta_2)$ to fuzzify p.d.f. as $f(\widetilde{D})$ in Section 3.4

When the p.d.f. of the random cash demand D_{ran} is $f(D) = \frac{1}{\theta} \exp\left(-\frac{D}{\theta}\right)$, the fuzzy case of Theorem 2 can be obtained. From Figs. 2–5, if $\Delta_1 = \Delta_2 = \Delta \to 0$, then the fuzzy case is close to the crisp case of Theorem 3.

5.2. The reason for the application of metric ρ in Definition 6

In Theorem 1 of Section 2 provided by Goetschel and Voxman (1986), two r-level sets $([a(r,x), b(r,x)], [p(r,x), q(r,x)])$ of intervals are regarded as a point in two-dimensional vector space. That is to apply the operations $(+, -, \times)$ of vectors and to use the metric D_1 in two-dimensional vector space to demonstrate Theorem 1. But in this paper, the operations $(+, -, \times)$ of crisp intervals shown in (10) in Section 3.3 are not the vectors operations. The operations $((+), (-), (\times))$ of fuzzy sets are to apply the operations $(+, -, \times)$ of crisp intervals and decomposition theorem to find the operational results of fuzzy sets, and then the fuzzy integral in Property 4 will be derived from using the operations $((+), (-), (\times))$ of fuzzy sets to deal with metric ρ. Thus it can be seen that Theorem 1 provided by Goetschel and Voxman (1986) is due to using the operations of vectors but not the fuzzy intervals derived from using fuzzy operations and metric D_1. Namely, Theorem 1 does not satisfy the fuzzy principle. However, Property 4 is analogous with Theorem 1.

Furthermore, general metrics in two-dimensional vector space usually use the vectors' operations $(+, -, \times)$. It is not suitable to employ such metrics to find the fuzzy integral. The derivation of metric ρ used in our paper is to apply Definition 6 and the linear order d (see Properties 1 and 2) defined on F_s (i.e., the family of fuzzy sets on $\Re$ defined in Section 2) to operate our model and such metric ρ satisfies three metric axioms (e.g., see Property 3).

6. Concluding remarks

In practice, since most single-period cash management problems have no historical data to determine the cash demand and formulate its probability distribution function for computing the optimal cash-raising amount, the fuzzy model developed in this study should be more suitable to solve the real world cash management problems than non-fuzzy models. In this paper, we have focused on applying the fuzzy stochastic single-period model based on fuzzy integral method to solve the problem of cash management, where demand is regarded as hybrid data. In conclusion, we have explained why past data cannot sufficiently predict real cash demand that may depend on unexpected events. Our main contribution is to provide the analytical development of a theoretical model, so as to reveal the practicability of applying single-period inventory theory in the issue of cash management with the fuzzy sense.

Acknowledgement

The authors would like to thank the two anonymous referees for their constructive comments.

Appendix A. Proofs of Theorems

Proof of Theorem 2

(a) By Formula 1(a).
(b) To fuzzify the cash demand D of p.d.f. $f(D)$, we obtain a fuzzy set $f(\widetilde{D})$, and then utilize extension principle to find its membership function $\mu_{f(\widetilde{D})}(z)$ as follows:

Let $z = f(x)$, i.e., $x = -\theta \ln \theta z$, by (8), we have

$$\mu_{f(\widetilde{D})}(z) = \sup_{x=-\theta\ln\theta z} \mu_{\widetilde{D}}(x) = \begin{cases} \frac{D+\Delta_2+\theta\ln\theta+\theta\ln z}{\Delta_2}, & \frac{1}{\theta}\exp\left(-\frac{D+\Delta_2}{\theta}\right) \leqslant z \leqslant \frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right), \\ \frac{-\theta\ln\theta-D+\Delta_1-\theta\ln z}{\Delta_1}, & \frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right) \leqslant z \leqslant \frac{1}{\theta}\exp\left(-\frac{D-\Delta_1}{\theta}\right), \\ 0, & \text{otherwise}. \end{cases} \tag{A.1}$$

Thus, the left and right end points of the α-level $f(\widetilde{D})(\alpha)$ of $f(\widetilde{D})$ are

$$f(\widetilde{D})_{\mathrm{l}}(\alpha) = \exp\left(-\frac{D+\theta\ln\theta+(1-\alpha)\Delta_2}{\theta}\right); \tag{A.2}$$

$$f(\widetilde{D})_{\mathrm{r}}(\alpha) = \exp\left(-\frac{D+\theta\ln\theta-(1-\alpha)\Delta_1}{\theta}\right). \tag{A.3}$$

We also fuzzify $g(D)$ as $g(\widetilde{D}) = \widetilde{D}(\cdot)f(\widetilde{D})$, then from (8), $0 \leqslant D_{\mathrm{l}}(\alpha) < D_{\mathrm{r}}(\alpha)$ and $0 \leqslant f(\widetilde{D})_{\mathrm{l}}(\alpha) < f(\widetilde{D})_{\mathrm{r}}(\alpha)$, $\forall \alpha \in [0,1]$. By (10), the α-level set of $g(\widetilde{D})$ can be rewritten as

$$g(\widetilde{D})(\alpha) = [g(\widetilde{D})_{\mathrm{l}}(\alpha), g(\widetilde{D})_{\mathrm{r}}(\alpha)] = [D_{\mathrm{l}}(\alpha)f(\widetilde{D})_{\mathrm{l}}(\alpha), D_{\mathrm{r}}(\alpha)f(\widetilde{D})_{\mathrm{r}}(\alpha)]. \tag{A.4}$$

From Property 4 and (12), (13), we can obtain the fuzzy total cost $\widetilde{E}(R)$ in (11). Using signed distance method in Definition 4 to defuzzify (11), we have

$$E^{**}(R;\theta) \equiv d(\widetilde{E}(R), \widetilde{0}) = \frac{1}{2}\int_0^1 [\widetilde{E}(R)_{\mathrm{l}}(\alpha) + \widetilde{E}(R)_{\mathrm{r}}(\alpha)]\,\mathrm{d}\alpha = \frac{1}{2}[E^{**}(R)_{\mathrm{l}} + E^{**}(R)_{\mathrm{r}}], \tag{A.5}$$

where $\widetilde{E}(R)_{\mathrm{l}}(\alpha)$ and $\widetilde{E}(R)_{\mathrm{r}}(\alpha)$ are in (12) and (13) respectively,

$$E^{**}(R)_{\mathrm{l}} = \int_0^1 \widetilde{E}(R)_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha$$
$$= C(R-I) + HR\int_0^R \int_0^1 f(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D - H\int_0^R \int_0^1 g(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D$$
$$+ V\int_R^\infty \int_0^1 g(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D - VR\int_R^\infty \int_0^1 f(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D; \qquad \text{(A.6)}$$

$$E^{**}(R)_{\mathrm{r}} = \int_0^1 \widetilde{E}(R)_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha$$
$$= C(R-I) + HR\int_0^R \int_0^1 f(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D - H\int_0^R \int_0^1 g(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D$$
$$+ V\int_R^\infty \int_0^1 g(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D - VR\int_R^\infty \int_0^1 f(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}D. \qquad \text{(A.7)}$$

The integrals $\int_0^1 f(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha$, $\int_0^1 f(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha$, $\int_0^1 g(\widetilde{D})_{\mathrm{l}}(\alpha)\,\mathrm{d}\alpha$, and $\int_0^1 g(\widetilde{D})_{\mathrm{r}}(\alpha)\,\mathrm{d}\alpha$ must be calculated beforehand in order to solve (14) (or (A.6) and (A.7)). From (A.2)–(A.4), we have the following: for $j = 1, 2$,

$$L(D;j) \equiv \int_0^1 \exp\left(-\frac{D+\theta\ln\theta+(-1)^j(1-\alpha)\Delta_j}{\theta}\right)\mathrm{d}\alpha$$
$$= \frac{\theta}{(-1)^{j+1}\Delta_j}\int_{\frac{(D+\theta\ln\theta)+(-1)^j\Delta_j}{\theta}}^{\frac{(D+\theta\ln\theta)}{\theta}} \exp(-z)\,\mathrm{d}z \qquad \text{(A.8)}$$
$$= \frac{\theta}{(-1)^{j+1}\Delta_j}\left[\exp\left(\frac{(-1)^{j+1}\Delta_j}{\theta}\right)-1\right]\exp\left(-\frac{D+\theta\ln\theta}{\theta}\right).$$

Afterwards, we use the formula, $\int_a^b z(\exp(-z))\,\mathrm{d}z = \int_a^b z d(-\exp(-z)) = [-z(\exp(-z))]|_a^b + \int_a^b \exp(-z)\,\mathrm{d}z$, in the following operation: For $i, j = 1, 2$ and $i \neq j$,

$$K(D;i,j) \equiv \int_0^1 [D+(-1)^i(1-\alpha)\Delta_i]\exp\left(-\frac{D+\theta\ln\theta+(-1)^j(1-\alpha)\Delta_j}{\theta}\right)\mathrm{d}\alpha$$
$$= -\frac{\theta}{\Delta_j^2}\int_{\frac{(D+\theta\ln\theta+(-1)^j\Delta_j)}{\theta}}^{\frac{(D+\theta\ln\theta)}{\theta}}[(-1)^i\Delta_i\theta z + (-1)^j\Delta_j D - (-1)^i\Delta_i(D+\theta\ln\theta)]\exp(-z)\,\mathrm{d}z$$
$$= \frac{\Delta_i\theta}{\Delta_j}\exp\left(-\frac{D+\theta\ln\theta+(-1)^j\Delta_j}{\theta}\right)$$
$$+ \left[\frac{(-1)^{i+1}\Delta_i\theta^2}{\Delta_j^2} + \frac{(-1)^{j+1}\theta}{\Delta_j}D\right]\frac{(-1)^{j+1}\Delta_j}{\theta}L(D;j). \qquad \text{(A.9)}$$

Continuously, let

$$F_*(0,R;j) \equiv \int_0^R L(D;j)\,\mathrm{d}D = \frac{(-1)^{j+1}\theta}{\Delta_j}\left[\exp\left(\frac{(-1)^{j+1}\Delta_j}{\theta}\right)-1\right]\left[1-\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right)\right], \qquad \text{(A.10)}$$

$$F_*(R,\infty;j) \equiv \int_R^\infty L(D;j)\,\mathrm{d}D = \frac{(-1)^{j+1}\theta^2}{\Delta_j}\left[\exp\left(\frac{(-1)^{j+1}\Delta_j}{\theta}\right)-1\right]\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right), \qquad \text{(A.11)}$$

$$G_*(0,R;i,j) \equiv \int_R^\infty K(D;i,j)\,\mathrm{d}D$$
$$= \frac{\Delta_i\theta^2}{\Delta_j}\left[1-\exp\left(-\frac{R}{\theta}\right)\right]\exp\left(-\frac{\theta\ln\theta+(-1)^j\Delta_j}{\theta}\right)-\frac{\Delta_i\theta}{\Delta_j}F_*(0,R;j)$$
$$+\frac{(-1)^{j+1}\theta^2}{\Delta_j}\left[\exp\left(\frac{(-1)^{j+1}\Delta_j}{\theta}\right)-1\right]\left[1-(R+\theta)\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right)\right], \tag{A.12}$$

$$G_*(R,\theta;i,j) \equiv \int_R^\infty K(D;i,j)\,\mathrm{d}D$$
$$= \frac{\Delta_i\theta^2}{\Delta_j}\exp\left(-\frac{R+\theta\ln\theta+(-1)^j\Delta_j}{\theta}\right)-\frac{\Delta_i\theta}{\Delta_j}F_*(R,\infty;j)$$
$$+\frac{(-1)^{j+1}\theta^2}{\Delta_j}\left[\exp\left(\frac{(-1)^{j+1}\Delta_j}{\theta}\right)-1\right]\left[(R+\theta)\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right)\right]. \tag{A.13}$$

Hence, from (A.5)–(A.9), we have

$$E^{**}(R;\theta) = C(R-I)+\frac{1}{2}HR\int_0^R(L(D;1)+L(D;2))\,\mathrm{d}D$$
$$-\frac{1}{2}H\left(\int_0^R(K(D;1,2)+K(D;2,1)\right)\mathrm{d}D$$
$$+\frac{1}{2}V+\left(\int_R^\infty(K(D;1,2)+K(D;2,1)\right)\mathrm{d}D-\frac{1}{2}VR\int_R^\infty(L(D;1)+L(D;2)\,\mathrm{d}D. \tag{A.14}$$

In addition, from (A.10)–(A.14), the estimate of total cost in the fuzzy sense can be expressed as

$$E^{**}(R;\theta) = C(R-I)+\frac{1}{2}HR[F_*(0,R;1)+F_*(0,R;2)]-\frac{1}{2}H[G_*(0,R;1,2)+(G_*(0,R;2,1)]$$
$$+\frac{1}{2}V+[G_*(R,\infty;1,2)+G_*(R,\infty;2,1)]-\frac{1}{2}VR[F_*(R,\infty;1)+F_*(R,\infty;2)]. \tag{A.15}$$

To find the optimal solution, we take the first-order differential of $E^{**}(R;\theta)$ (in (A.14)) with respect to R, yields

$$\frac{\mathrm{d}}{\mathrm{d}R}E^{**}(R;\theta) = C+\frac{1}{2}H\int_0^R(L(D;1)+L(D;2))\,\mathrm{d}D+\frac{1}{2}HR[L(R;1)+L(R;2)]$$
$$-\frac{1}{2}H[K(R;1,2)+K(R;2.1)]-\frac{1}{2}V[K(R;1,2)+K(R;2,1)]$$
$$-\frac{1}{2}V\int_R^\infty(L(D;1)+L(D;2))+\frac{1}{2}VR[L(R;1)+L(R;2)]$$
$$= C+\frac{1}{2}H[F_*(0,R;1)+F_*(0,R;2)]-\frac{1}{2}V[F_*(R,\infty;1)+F_*(R,\infty;2)]$$
$$+\frac{1}{2}(H+V)[(RL(R;1)-K(R;2,1))+(RL(R;2)-K(R;1,2))]. \tag{A.16}$$

Using (A.8) and (A.9) to compute the last term on the right-hand side of (A.16), yields

$$RL(R;j) - K(R;i,j) = -\left[\frac{\varDelta_i\theta}{\varDelta_j}.\exp\left(\frac{(-1)^{j+1}\varDelta_j}{\theta}\right) + \frac{(-1)^{i+1}\varDelta_i\theta^2}{\varDelta_j^2}\left(\exp\left(\frac{(-1)^{j+1}\varDelta_j}{\theta}\right) - 1\right)\right] \times \exp\left(-\frac{R+\theta\ln\theta}{\theta}\right). \quad (A.17)$$

Then, let

$$A(\theta,\varDelta_1,\varDelta_2) = C + \frac{\theta}{2}H\left[\frac{1}{\varDelta_1}\left(\exp\left(\frac{\varDelta_1}{\theta}\right) - 1\right) - \frac{1}{\varDelta_2}\left(\exp\left(-\frac{\varDelta_2}{\theta}\right) - 1\right)\right],$$

$$\begin{aligned}B(\theta,\varDelta_1,\varDelta_2) = \frac{1}{2}(H+V)\Bigg[&\frac{\theta^2}{\varDelta_1}\left(\exp\left(\frac{\varDelta_1}{\theta}\right) - 1\right) - \frac{\theta^2}{\varDelta_2}\left(\exp\left(-\frac{\varDelta_2}{\theta}\right) - 1\right)\\ &+ \frac{\varDelta_1\theta}{\varDelta_2}\exp\left(-\frac{\varDelta_2}{\theta}\right) + \frac{\varDelta_1\theta^2}{\varDelta_2^2}\left(\exp\left(-\frac{\varDelta_2}{\theta}\right) - 1\right) + \frac{\varDelta_2\theta}{\varDelta_1}\exp\left(-\frac{\varDelta_1}{\theta}\right)\\ &- \frac{\varDelta_2\theta^2}{\varDelta_1^2}\left(\exp\left(+\frac{\varDelta_1}{\theta}\right) - 1\right)\Bigg].\end{aligned} \quad (A.18)$$

Substituting (A.17) and (A.18) into (A.16) and taking the second derivative of $E^{**}(R;\theta)$ with respect to R yields

$$\frac{d}{dR}E^{**}(R;\theta) = A(\theta,\varDelta_2,\varDelta_2) - B(\theta,\varDelta_1,\varDelta_2)\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right),$$

and

$$\frac{d^2}{dR^2}E^{**}(R;\theta) = \frac{B(\theta,\varDelta_1,\varDelta_2)}{\theta}\exp\left(-\frac{R+\theta\ln\theta}{\theta}\right) > 0, \quad \text{if } B(\theta,\varDelta_1,\varDelta_2) > 0.$$

Hence, taking $\frac{d}{dR}E^{**}(R;\theta) = 0$ and if $B(\theta,\varDelta_1,\varDelta_2) > 0$, $A(\theta,\varDelta_1,\varDelta_2) > 0$, $B(\theta,\varDelta_1,\varDelta_2) > \theta A(\theta,\varDelta_1,\varDelta_2)$, then we have the following optimal cash-raising amount in the fuzzy sense given by

$$R = \theta\ln\frac{B(\theta,\varDelta_1,\varDelta_2)}{\theta A(\theta,\varDelta_1,\varDelta_2)} \equiv R_2^*(\theta;\varDelta_1,\varDelta_2), \quad \text{where } 0 < \theta \leqslant 1.$$

Also, the minimum total cost $E^{**}(R_2^*(\theta;\varDelta_1,\varDelta_2);\theta)$ can be obtained by substituting $R_2^*(\theta)$ into R of $E^{**}(R;\theta)$, which is shown in (A.15). □

Proof of Theorem 3.

By $\int_0^{R_1^*(\theta)}\frac{1}{\theta}\exp\left(-\frac{D}{\theta}\right)dD = \frac{V-C}{H+V}$, if $V > C$, have (a). Substituting the result of (a), $\exp\left(-\frac{R_1^*(\theta)}{\theta}\right) = \frac{H+C}{H+V}$, into (6), Theorem 3 holds. □

References

Asmussen, S., Perry, D., 1998. An operational calculus for matrix-exponential distributions with an application to a Brownian (q,Q) inventory problem. Mathematics of Operations Research 23, 166–176.

Asmussen, S., Taksar, M.I., 1997. Controlled diffusion models for optimal dividend pay-out. Insurance: Mathematics and Economics 20, 1–15.