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Valuation by using a fuzzy discounted cash flow model

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Abstract

The objective of this paper is to extend the classical discounted cash flow (DCF) model by developing a fuzzy logic system that takes vague cash flow and imprecise discount rate into account. In order to explicitly discuss a more appropriate valuation model, uncertain information will be fuzzified as triangular fuzzy numbers to quantify and evaluate the intrinsic value of a company's financial asset under the framework of DCF approach. We will find that the fuzzy discounted cash flow (FDCF) model proposed in this paper is one extension of classical (crisp) model and should be more suitable to capture the elements of valuation than non-fuzzy models.

Keywords: Valuation; Fuzzy number; Discounted cash flow (DCF); λ-signed distance method

1. Introduction

A clear thinking about valuation and skill in using a right valuation method to guide business decisions are prerequisites for success in current competitive environment. Generally speaking, all management decisions are based on some valuation model. It is therefore to the managers' advantage to base their decisions on the model that most accurately reflects company value. The discounted cash flow (DCF) model, an economic model, studied by the classical mathematics of finance describes some very general ways to characterize the expression of its present value (see, e.g. Brigham, 1992; Sharpe, Alexander, & Bailey, 1999). In practice, the DCF model has become very popular in valuation because it is most consistent with the goal of long-term value creation, and it may capture all the elements that affect the value of the company in a comprehensive yet straightforward manner. It is also widely applied in many fields such as project management, insurance, and financial management. Some proponents of the DCF approach have even suggested that the DCF model can provide a more sophisticated and reliable picture of a company's value than the accounting approach (Copeland, Koller, & Murrin, 1994). Furthermore, the DCF approach is strongly supported by research into how the stock markets actually value companies since the stock is one of the financial assets that take the dividend paying to be the primary source of cash flow. In this way, if a investor is able to accurately estimate the future cash-flow stream of a financial asset and to match up an appropriate discount rate, then they would easily compute the fair asset's value, so as to take an action (to sell or continuing to hold the asset).

However, such a classical DCF model does not incorporate the uncertainties, which may be inherent in the parameters used in it. Because various types of uncertainties and imprecision such as discount rate and future cash flows are inherent in the financial environments, the uncertain parameters are usually regarded as a constant or treated as a random variable that may be estimated by past statistical data. In realistic situations, unfortunately, such the estimation is often biased. For example, Shiller (1981) applied DCF model to derive the upper and lower bounds of fluctuation of stock prices, but the empirical results showed that the real stock prices obviously went beyond the scope. In addition, although some existing literature has incorporated uncertainty into the related fields of investment decision based on intuitive methods

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or a probabilistic approach (see, e.g. Brigham, 1992; Hurley & Johnson, 1994; Liang & Song, 1994), and the uncertain information is therefore estimated by using educated guesses or other statistical techniques, there are the disadvantages of depending too much on the intuition of the decision maker and requiring the fulfillment of some assumptions about probabilistic distributions.

Recently, some developments in fuzzy-financial mathematics have been applied to the valuation issues. Buckley (1987) studied the fuzzy extension of the mathematics of finance to concentrate on the compound interest law. Then, Li Calzi (1990) investigated a possible general setting by considering both compact fuzzy intervals and invertible fuzzy intervals for the fuzzy mathematics of finance. Kuchta (2000) also generalized fuzzy equivalents for methods of evaluating investment projects. Furthermore, to observe the investors' behavior in the financial market with a complicated and uncertain environment, investors are always trying to rely on some ways to accurately predict the prices of a specific financial asset, but often have less than successful results. For this reason, several researchers endeavored to propose a series of excellent studies based on fuzzy techniques in order to valuate the stock market and further to predict stock prices accurately. For example, Dourra and Siy (2002) applied fuzzy information technologies to investments through technical analysis, and used them to examine various companies to achieve a substantial investment return. Kuo, Chen, and Hwang (2001) used genetic algorithm based on fuzzy neural networks to measure quantitative and qualitative effects on the stock market. Wang (2002) proposed a fuzzy grey prediction system to analyze stock data and to predict stock prices, and then he employed fuzzy rough set system to predict the stronger rules of stock price, achieving a higher degree of accuracy (see, e.g. Wang, 2003). Nevertheless, the models employed in their studies are much complicated and the ordinary investors' concerns still exist.

In view of the above, indeed, there is an opportunity to improve the classical DCF model by using the advances in mathematics and sciences of fuzzy set theory. Namely, fuzzy reasoning is very effective in such environments. Therefore, the purpose of this paper is to extend a classical (crisp) DCF model that can be fed with a fuzzy system, hoping to make it more applicable in practice. We start to describe the DCF model in its classical form. Further, all uncertain parameters will be given a fuzzified form in the paper.

The rest of this paper is organized as follows. Section 2 states the preliminaries where we define the λ-signed distance method which is similar to Yao and Wu (2000), and then employ it to formulate the fuzzy discount cash flow (FDCF) model. The crisp DCF model will be surveyed in Section 3 first. Then in Section 4, triangular fuzzy numbers and their operations will be performed and discussed with regard to the fuzzy valuation model. In Section 5, the results derived from the fuzzy case in Section 4 will be compared to that of crisp case with numerical operations, and then the implications of the FDCF model are discussed in Section 6. Finally, we give the conclusion remarks in Section 7.

2. Preliminaries

Before presenting the FDCF model based on the λ-signed distance method, the following definitions are provided in advance with some relevant operations (see, e.g. Kaufmann & Gupta, 1991).

Definition 2.1. A fuzzy set $[a,b;\alpha]$, $a<b$ defined on $\Re=(-\infty,\infty)$, which has the following membership function, is called a level α fuzzy interval.

$$\mu_{[a,b;\alpha]}(x)=\begin{cases}\alpha, & a\le x\le b,\\ 0, & \text{otherwise.}\end{cases}$$

Definition 2.2. By Pu and Liu (1980), fuzzy point $\tilde{a}$ is a fuzzy set defined on $\Re$ with the following membership function:

$$\mu_{\tilde{a}}(x)=\begin{cases}1, & x=a,\\ 0, & x\neq a.\end{cases}$$

Definition 2.3. The triangular fuzzy number $\tilde{B}$ is defined on $\Re$ with a membership function as follows, and denoted by $\tilde{B}=(a,b,c)$, where $a<b<c$.

$$\mu_{\tilde{B}}(x)=\begin{cases}\dfrac{x-a}{b-a}, & a\le x\le b,\\ \dfrac{c-x}{c-b}, & b\le x\le c,\\ 0, & \text{otherwise.}\end{cases}$$

Let F_s be the family of fuzzy sets defined on $\Re$, for each $\tilde{D}\in F_s$, the α-*cut* of $\tilde{D}$ is denoted by $D(\alpha)=\{x|\mu_{\tilde{D}}(x)\ge\alpha\}=[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$ $(0\le\alpha\le1)$, and both $\tilde{D}_L(0)$ and $\tilde{D}_U(0)$ are finite values. For each $\alpha\in[0,1]$, the real numbers $\tilde{D}_L(\alpha)$, $\tilde{D}_U(\alpha)$ separately represent the left and right end points of $D(\alpha)$ and satisfy the conditions that both of $\tilde{D}_L(\alpha)$, $\tilde{D}_U(\alpha)$ exist in $\alpha\in[0,1]$ and are continuous over [0,1].

Let $\tilde{D}\in F_s$, by decomposition theory, we have

$$\tilde{D}=\bigcup_{0\le\alpha\le1}\alpha I_{D(\alpha)},$$

where $I_{D(\alpha)}$ is the characteristic function of $D(\alpha)$. By Definition 2.1, if $x\in D(\alpha)$, then $\alpha I_{D(\alpha)}(x)=\alpha=\mu_{[D_L(\alpha),D_U(\alpha);\alpha]}(x)$, and if $x\notin D(\alpha)$, then $\alpha I_{D(\alpha)}(x)=0=\mu_{[D_L(\alpha),D_U(\alpha);\alpha]}(x)$, therefore, we have

$$\tilde{D}=\bigcup_{0\le\alpha\le1}\alpha I_{D(\alpha)}=\bigcup_{0\le\alpha\le1}[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha);\alpha]. \tag{2.1}$$

Introducing the concept of Yao and Wu (2000), we consider the signed distance and ranking on F_s and provide Definitions 2.4–2.6 as follows:

For each $\lambda\in(0,1)$, the λ-signed distance of closed interval $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$ from origin 0 can be defined by

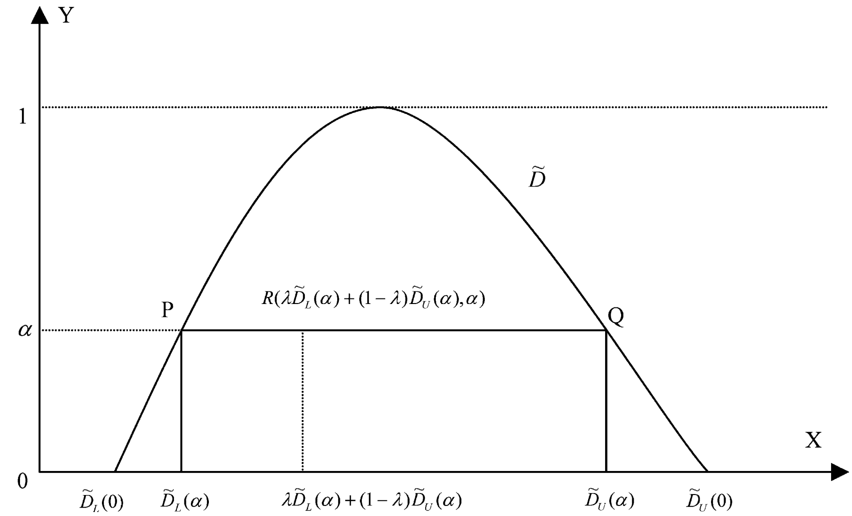

Fig. 1. α-*cut* $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$ and point $\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$ in $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$.

$d_0([\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)], 0; \lambda) = \lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)$, where $\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)$ is a inner point in $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)]$. (See Fig. 1.)

Since for each $\alpha \in [0,1]$, $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)] \leftrightarrow [\tilde{D}_L(\alpha), \tilde{D}_U(\alpha); \alpha]$ and $0 \leftrightarrow \tilde{0}$ are one-to-one mapping, the λ-signed distance of $[\tilde{D}_L(\alpha), \tilde{D}_U(\alpha); \alpha]$ from $\tilde{0}$ can be defined by $d([\tilde{D}_L(\alpha), \tilde{D}_U(\alpha); \alpha], \tilde{0}; \lambda) = d_0([\tilde{D}_L(\alpha), \tilde{D}_U(\alpha)], 0; \lambda) = \lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)$.

For each $\tilde{D} \in F_s$ $(0 \le \alpha \le 1)$, $\tilde{D}_L(\alpha)$, $\tilde{D}_U(\alpha)$ is a function of α and continuous over [0,1], so the integral mean value of the λ-signed distance is

$$\int_0^1 d([\tilde{D}_L(\alpha)\tilde{D}_U(\alpha); \alpha], \tilde{0}; \lambda) d\alpha$$

$$= \int_0^1 (\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)) d\alpha. \tag{2.2}$$

According to (2.1) and (2.2), we have the Definition 2.4 as follows.

Definition 2.4. (a) For each $\tilde{D} \in F_s$ and each $\lambda \in (0,1)$, the λ-signed distance from $\tilde{D}$ to $\tilde{0}$ is defined by

$$d(\tilde{D}, \tilde{0}; \lambda) = \int_0^1 [\lambda\tilde{D}_L(\alpha) + (1-\lambda)\tilde{D}_U(\alpha)] d\alpha.$$

(b) When $\tilde{D} = (a, a, a) = \tilde{a}$ is a fuzzy point at α and for all $\alpha \in [0,1]$, $\tilde{D}_L(\alpha) = \tilde{D}_U(\alpha) = a$, then $d(\tilde{a}, \tilde{0}; \lambda) = a$ for all $\lambda \in (0,1)$.

Next, the arithmetic operations of level α fuzzy intervals $\alpha \in [0,1]$ are shown below:

$$[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha); \alpha](+)[\tilde{B}_L(\alpha), \tilde{B}_U(\alpha); \alpha]$$

$$= [\tilde{A}_L(\alpha) + \tilde{B}_L(\alpha), \tilde{A}_U(\alpha) + \tilde{B}_U(\alpha); \alpha]. \tag{2.3}$$

When $0 \le \tilde{A}_L(\alpha) < \tilde{A}_U(\alpha)$ and $0 \le \tilde{B}_L(\alpha) < \tilde{B}_U(\alpha)$, we have

$$[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha); \alpha](\times)[\tilde{B}_L(\alpha), \tilde{B}_U(\alpha); \alpha]$$

$$= [\tilde{A}_L(\alpha)\tilde{B}_L(\alpha), \tilde{A}_U(\alpha)\tilde{B}_U(\alpha); \alpha]. \tag{2.4}$$

Similarly, when $0 \le \tilde{A}_L(\alpha) < \tilde{A}_U(\alpha)$ and $0 < \tilde{B}_L(\alpha) < \tilde{B}_U(\alpha)$, we also have

$$[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha); \alpha](\div)[\tilde{B}_L(\alpha), \tilde{B}_U(\alpha); \alpha]$$

$$= \left[\frac{\tilde{A}_L(\alpha)}{\tilde{B}_U(\alpha)}, \frac{\tilde{A}_U(\alpha)}{\tilde{B}_L(\alpha)}; \alpha\right]. \tag{2.5}$$

Additionally,

$$\tilde{k}(\times)[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha); \alpha] = \begin{cases} [k\tilde{A}_L(\alpha), k\tilde{A}_U(\alpha); \alpha] & \text{if } k > 0, \\ [k\tilde{A}_U(\alpha), k\tilde{A}_L(\alpha); \alpha] & \text{if } k < 0. \end{cases} \tag{2.6}$$

Definition 2.5. Let $\tilde{A}$, $\tilde{B} \in F_s$ and for each $\lambda \in (0,1)$, define the metric ρ_λ by

$$\rho_\lambda(\tilde{A}, \tilde{B}) = |d(\tilde{A}, \tilde{0}; \lambda) - d(\tilde{B}, \tilde{0}; \lambda)|.$$

Definition 2.6. For $\tilde{A}$, $\tilde{B} \in F_s$ and each $\lambda \in (0,1)$, relations '$\prec$, $\approx$' on F_s are defined by

$$\tilde{A} \prec \tilde{B} \text{ iff } d(\tilde{A}, \tilde{0}; \lambda) < d(\tilde{B}, \tilde{0}; \lambda);$$

$$\tilde{A} \approx \tilde{B} \text{ iff } d(\tilde{A}, \tilde{0}; \lambda) = d(\tilde{B}, \tilde{0}; \lambda).$$

Using Definition 2.6 and the ordering relations $<$, $=$ defined on $\Re$, then the following Properties 2.7 and 2.8 can be proved.

Property 2.7. *For*$\tilde{A}$ $\tilde{B} \in F_s$ *and each* $\lambda \in (0,1)$*, the ordering relations* $\prec$, $\approx$ *defined on* F_s *satisfy the law of trichotomy.*

Namely, one and only one of the three relations of $\tilde{A} \prec \tilde{B}$, $\tilde{A} \approx \tilde{B}$, $\tilde{B} \prec \tilde{A}$ *must hold.*

Property 2.8. *For* $\tilde{A}$, $\tilde{B}$, $\tilde{C} \in F_s$ *and each* $\lambda \in (0,1)$, *the ordering relations* $\prec$, $\approx$ *defined on* F_s *satisfy the following axioms*:

(1°) $\tilde{A} \prec \approx \tilde{A}$;
(2°) $\tilde{A} \prec \approx \tilde{B}$, and $\tilde{B} \prec \approx \tilde{A}$, then $\tilde{A} \approx \tilde{B}$;
(3°) $\tilde{A} \prec \approx \tilde{B}$, and $\tilde{B} \prec \approx \tilde{C}$, then $\tilde{A} \prec \approx \tilde{C}$.

From Properties 2.7 and 2.8, we know that the ordering relations $\prec$, $\approx$ on F_s are linear order.

Property 2.9. *For* $\tilde{A}$, $\tilde{B}$, $k \in F_s$ *and each* $\lambda \in (0,1)$, *the following two characteristics hold*:

(1°) $d(\tilde{A}(+)\tilde{B}, \tilde{0}; \lambda) = d(\tilde{A}, \tilde{0}; \lambda) + d(\tilde{B}, \tilde{0}; \lambda)$;
(2°) $d(k\tilde{A}, \tilde{0}; \lambda) = k\, d(\tilde{A}, \tilde{0}; \lambda)$.

Proof. (1°) By (2.3), we know

$$\tilde{A}(+)\tilde{B} = \bigcup_{0 \le \alpha \le 1} [\tilde{A}_L(\alpha) + \tilde{B}_L(\alpha), \tilde{A}_U(\alpha) + \tilde{B}_U(\alpha); \alpha],$$

then by Definition 2.4,

$$\begin{aligned} & d(\tilde{A}(+)\tilde{B}, \tilde{0}; \lambda) \\ & \quad = \int_0^1 [\lambda(\tilde{A}_L(\alpha) + \tilde{B}_L(\alpha)) + (1-\lambda)(\tilde{A}_U(\alpha) + \tilde{B}_U(\alpha))] \mathrm{d}\alpha \\ & \quad = \mathrm{d}(\tilde{A}, \tilde{0}; \lambda) + \mathrm{d}(\tilde{B}, \tilde{0}; \lambda). \end{aligned}$$

(2°) By (2.6) and Definition 2.4, $d(k\tilde{A}, \tilde{0}; \lambda) = k\, d(\tilde{A}, \tilde{0}; \lambda)$ is proved. □

Property 2.10. *For* $\tilde{A}$, $\tilde{B}$, $\tilde{C} \in F_s$ *and each* $\lambda \in (0,1)$, *metric* ρ_λ *satisfies the following three metric axioms*:

(1°) $\rho_\lambda(\tilde{A}, \tilde{B}) = 0$ iff $\tilde{A} \approx \tilde{B}$;
(2°) $\rho_\lambda(\tilde{A}, \tilde{B}) \ge 0$;
(3°) $\rho_\lambda(\tilde{A}, \tilde{B}) + \rho_\lambda(\tilde{B}, \tilde{C}) \ge \rho_\lambda(\tilde{A}, \tilde{C})$.

Proof. (1°) and (2°) can be proved by Definition 2.6 and the characters of the ordering relations $<$, $=$ defined on $\Re$. Because

$$\begin{aligned} & \rho_\lambda(\tilde{A}, \tilde{B}) + \rho_\lambda(\tilde{B}, \tilde{C}) \\ & \quad = |d(\tilde{A}, \tilde{0}; \lambda) - d(\tilde{B}, \tilde{0}; \lambda)| + |d(\tilde{B}, \tilde{0}; \lambda) - d(\tilde{C}, \tilde{0}; \lambda)| \\ & \quad \ge |d(\tilde{A}, \tilde{0}; \lambda) - d(\tilde{C}, \tilde{0}; \lambda)| \\ & \quad = \rho_\lambda(\tilde{A}, \tilde{C}), \text{ (3°) holds.} \end{aligned}$$ □

Remark 2.11. By Property 2.10, for each $\lambda \in (0,1)$, (ρ_λ, F_s) is a metric space in the fuzzy sense.

Remark 2.12. Let $F_P = \{\tilde{a} | a \in \Re\}$ be the family of all fuzzy points on $\Re = (-\infty, \infty)$. Obviously, $F_P \subset F_s$, by Definitions 2.4 and 2.5, when $\tilde{a}$, $\tilde{b} \in F_P$, for each $\lambda \in (0,1)$, we have $d(\tilde{a}, \tilde{0}; \lambda) = a$, $d(\tilde{b}, \tilde{0}; \lambda) = b$, and $\rho_\lambda(\tilde{a}, \tilde{b}) = |a-b|$. When $a, b \in \Re$, by Definition 2.4, we have $d_0(a,0) = a$, $d_0(b,0) = b$, and $\rho_0(a,b) = |d_0(a,0) - d_0(b,0)| = |a-b|$. Thus, ρ_0 is the metric function on $\Re$. Meanwhile, we know that $\tilde{a} \in F_P \leftrightarrow a \in \Re$ is one-to-one mapping from F_P to $\Re$ and satisfies the following relations:

(1°) For each $\lambda \in (0,1)$, $d(\tilde{a}, \tilde{0}; \lambda) = a = d_0(a, 0)$, $\rho_\lambda(\tilde{a}, \tilde{b}) = |a-b| = \rho_0(a,b)$, for all a, $b \in \Re$.
(2°) For each $\lambda \in (0,1)$, the relation of three metric spaces $(\Re, \rho_0)$, (F_P, ρ_λ), (F_s, ρ_λ) is $(\Re, \rho_0) \equiv (F_P, \rho_\lambda) \subset (F_s, \rho_\lambda)$. That is, metric space (F_s, ρ_λ) is one extension of real metric space $(\Re, \rho_0)$.

3. Valuation by using the crisp DCF model

The crisp DCF model is a well-known approach to valuation (see, e.g. Brigham, 1992; Copeland et al., 1994; Sharpe et al., 1999), whereby estimated future cash flows are 'discounted' at an interest rate (also called: 'rate of return'), that reflects the perceived riskiness of the cash flows. The discount rate reflects two things: one is the time value of the money (investors would rather have cash immediately than having to wait and must therefore be compensated by paying for the delay); the other is the risk premium that reflects the extra return investors demand because they want to be compensated for the risk that the cash flow might not materialize after all. In other words, the valuation based on the DCF approach is the future expected cash flow discounted at a rate that reflects the riskiness of the cash flow.

The generalized DCF model such as Fig. 2 describes the return of a financial asset offering the investor a cash-flow stream, and today's intrinsic value is calculated as the present value of an infinite cash-flow stream.

From Fig. 2, the present value at time t for each time $t+j$, $j = 1,2,\ldots$ can be expressed as:

$$A_{t+1} = \frac{D_{t+1}}{1+k_{t+1}},$$

$$A_{t+2} = \frac{D_{t+2}}{(1+k_{t+1})(1+k_{t+2})}, \cdots,$$

$$A_{t+T} = \frac{D_{t+T}}{(1+k_{t+1})(1+k_{t+2}), \cdots, (1+k_{t+T})}, \cdots.$$

Therefore, the intrinsic value in a crisp model can now be expressed as the present value of expected future cash flows at time t that is given by

$$V_t^* = \sum_{i=1}^{n} \left[\prod_{j=1}^{i} \frac{1}{1+k_{t+j}} \right] D_{t+i}, \quad t = 0, 1, 2, \ldots, \tag{3.1}$$

where

n: life of the asset.
V_t^*: the intrinsic value at time t of an infinite cash-flow stream.

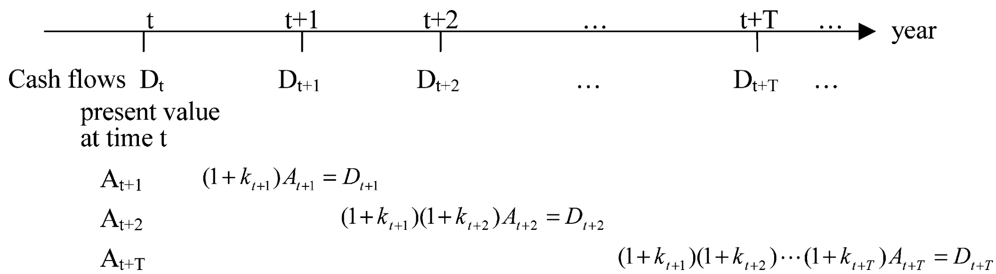

Fig. 2. Infinite cash-flow stream and present value at time t.

$D_{t+\mathrm{i}}$: cash flow in period t.
$k_{t+\mathrm{j}}$ discount rate at time $t+j$, reflecting the riskiness of the estimated cash flows; or required rate of return, the investor considers the returns available on other investments.

Formula (3.1) is a generalized DCF model in the sense that the time pattern of $D_{t+\mathrm{i}}$ should be a non-negative real number. It means that $D_{t+\mathrm{i}}$ may be rising, falling, or constant, or it may be fluctuated randomly. Generally speaking, when the future cash flows of a company or an asset follow a systematic pattern, some extensions and applications from the basic model can be derived easily. For example, the dividend discount model is a specialized case of equity valuation and the value of a stock is the present value of expected future dividends.

In real economic environment, the companies go through life cycle. Such like some high-tech companies, their operation usually have the following pattern with regard to the economic cycle: during the early part of their lives, the company's growth rate is higher than the average level of the economy's growth; then match the economy's growth; and finally lower than that of the economy's growth. In other words, the cash flow (D) of a specific asset for each year should depend on the individual company's growth rate (g). If the discount rate in each period is supposed to equalize, meaning $k=k_1=k_2=\cdots=k_{\mathrm{T}}=\cdots$, and k is a positive real number, then (3.1) can be simplified as

$$V_0^* = \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t}, \tag{3.2}$$

where $D_t=D_{t-1}(1+g_t)$. It means that D_t is a non-negative random variable with respect to g_t. In practice, unfortunately, the investor cannot use (3.2) to calculate the intrinsic value in its present form through crisp convergent formula, so we assume that most of the investors would eventually finance the asset at a default value after holding it for n years. Namely, the specific asset has uncertain cash flows during a certain time period n and finally sold at price P_n. Hence, (3.2) can be modified by

$$V_0^* = \sum_{t=1}^{n} \frac{D_t}{(1+k)^t} + \frac{P_n}{(1+k)^n} \tag{3.3}$$

where $D_t=D_{t-1}(1+g_t)$, $t=1,2,\ldots,n$. Note that (3.3) is usually regarded as a valuation model for the non-constant growth stocks. Similarly, if $g_t=0$ (i.e. fixed cash flows D_t such as the payments of interests) and k, P_n are, respectively regarded as the yields-to-maturity (YTM) and par value, then (3.3) becomes a classical bond valuation model.

In this case, if $g=g_1=g_2=\cdots=g_{\mathrm{n}}=\cdots$, and

$$\lim_{n\to\infty} P_n = 0,$$

then we can get the special case of (3.3) as below:

$$V_0^* = \lim_{n\to\infty}\left(\sum_{t=1}^{n} \frac{D_t}{(1+k)^t} + \frac{P_n}{(1+k)^n}\right)$$

$$= \lim_{n\to\infty}\sum_{t=1}^{n} \frac{D_0(1+g)^t}{(1+k)^t} = \frac{D_0(1+g)}{k-g}, \tag{3.4}$$

where $k>g$. That is so-called the Gordon Model (see, e.g. Gordon, 1962). Note that

$$\lim_{n\to\infty}\frac{1}{(1+k)^n} = 0,$$

and k is a positive real number. Moreover, if $g=0$, (3.4) can be simplified as

$$V_0^* = \lim_{n\to\infty}\left(\sum_{t=1}^{n} \frac{D_t}{(1+k)^t} + \frac{P_n}{(1+k)^n}\right) = \frac{D_0}{k}, \tag{3.5}$$

such a model is often used to evaluate the intrinsic value of *preferred stocks*. In other words, both (3.4) and (3.5) may be regarded as the special cases of (3.3).

However, when n is finite, it is quite difficult for typical investors to precisely predict the future price P_n at a certain value for a long-term period, so the present value V_0^* in (3.3) cannot be calculated directly. Also, it is difficult to suppose that the investors can predict a large number of g_n. That is why the general case is usually restricted to two or three stage models (Sorensen & Williamson, 1985).

4. Valuation by using the FDCF model

Due to the difficulties of precisely estimating the future cash flows, the discount rate (or the required rate of return), and the price at the nth year, the investors who apply the classical (crisp) DCF model to evaluate the intrinsic value of a specific asset often need to make several assumptions about the cash flow and the discount rate, and further to estimate them by using educated guesses or other statistical skills. For example, sometimes it is more realistic for the investor to additionally estimate these parameters in DCF model by linking the growth rate (g) with the other financial data such as ROE, P/E ratio or pay-out, instead estimating g directly (see, e.g. Leibowitz & Kogelman, 1994; Sorensen & Williamson, 1985).

On the other hand, regarding the acquisition of discount rate (k) (or the required rate of return), it is usually derived from the CAPM framework considering the risk factor, market expected rate of return, and expectations about the risk-free rate (Sharpe et al., 1999). However, since either the financial data or g are uncertain, these magnitudes should be more suitable to be directly considered as fuzzy numbers by fuzzifying g and k in order to simplify the arithmetic operations. Based on this, the fuzzy method defined in Section 2 is a more effective tool to evaluate the intrinsic value when future cash flows, discount rate, and growth rate cannot be precisely estimated as well as the risks. In this section, we will derive a FDCF model from Section 3 where the crisp DCF model has been surveyed.

First, we recall the crisp DCF model mentioned in (3.3) and let

$$G \equiv \sum_{t=1}^{n} \frac{D_t}{(1+k)^t} = \sum_{t=1}^{n} \frac{D_0(1+g_1)(1+g_2)\cdots(1+g_t)}{(1+k)^t}, \quad H \equiv \frac{P_n}{(1+k)^n}, \tag{4.1}$$

where $n \geq 1$, then (3.3) can be simplified to write:

$$V_0^* = G + H. \tag{4.2}$$

Next, we fuzzify D_0, g_j ($j=1,2,\ldots,t$), k and P_n as triangular fuzzy numbers $\tilde{D}_0$, $\tilde{g}_j$, $\tilde{k}$, and $\tilde{P}_n$ corresponding to the crisp intervals $[D_0-\omega_1, D_0+\omega_2]$, $[g_j-\omega_{j3}, g_j+\omega_{j4}]$, $[k-\omega_5, k+\omega_6]$, and $[P_n-\omega_7, P_n+\omega_8]$, respectively, and then yield

$$\tilde{D}_0 = (D_0-\omega_1, D_0, D_0+\omega_2); \tag{4.3}$$

$$\tilde{g}_j = (g_j-\omega_{j3}, g_j, g_j+\omega_{j4}), \quad j=1,2,\ldots,t; \tag{4.4}$$

$$\tilde{k} = (k-\omega_5, k, k+\omega_6); \tag{4.5}$$

$$\tilde{P}_n = (P_n-\omega_7, P_n, P_n+\omega_8), \tag{4.6}$$

where ω_1, ω_2, ω_{j3}, ω_{j4}, ω_5, ω_6, ω_7, and ω_8 may be appropriately determined by the decision maker to satisfy the following conditions:

$$0<\omega_1<D_0, \quad 0<\omega_2; \tag{4.7}$$

$$0<\omega_{j3}<g_j, \quad 0<\omega_{j4}, \ j=1,2,\ldots,t; \tag{4.8}$$

$$0<\omega_5<k, \quad 0<\omega_6; \tag{4.9}$$

$$0<\omega_7<P_n, \quad 0<\omega_8. \tag{4.10}$$

Using (4.3) to (4.6) to fuzzify (4.1), we have

$$\tilde{G} = \sum_{t=1}^{n} [\tilde{D}_0(\times)(\tilde{1}(+)\tilde{g}_1)(\times)(\tilde{1}(+)\tilde{g}_2)(\times)\cdots(\times) \times(\tilde{1}(+)\tilde{g}_t)](\div)(\tilde{1}(+)\tilde{k})^t, \tag{4.11}$$

$$\tilde{H} = \tilde{P}_n(\div)(\tilde{1}(+)\tilde{k})^n, \quad n \geq 1, \tag{4.12}$$

where the left and right end points of α-*cut* of $\tilde{D}_0$, $\tilde{g}_j$, $\tilde{k}$, and $\tilde{P}_n$ are

$$\begin{aligned} D_{0L}(\alpha) &= D_0-(1-\alpha)\omega_1(>0), \\ \tilde{D}_{0U}(\alpha) &= D_0+(1-\alpha)\omega_2(>0); \\ \tilde{g}_{jL}(\alpha) &= g_j-(1-\alpha)\omega_{j3}(>0), \\ \tilde{g}_{jU}(\alpha) &= g_j+(1-\alpha)\omega_{j4}(>0) \ j=1,2,\ldots,t; \\ \tilde{k}_L(\alpha) &= k-(1-\alpha)\omega_5(>0), \\ \tilde{k}_U(\alpha) &= k+(1-\alpha)\omega_6(>0); \\ \tilde{P}_{nL}(\alpha) &= P_n-(1-\alpha)\omega_7(>0), \\ \tilde{P}_{nU}(\alpha) &= P_n+(1-\alpha)\omega_8(>0). \end{aligned} \tag{4.13}$$

From (4.11) to (4.13), (2.3) to (2.6), the left and right end points of α-*cut* of $\tilde{G}$, $\tilde{H}$ can be written as

$$\begin{aligned} \tilde{G}_L(\alpha) &= \sum_{t=1}^{n} \frac{(D_0-(1-\alpha)\omega_1)(1+g_1-(1-\alpha)\omega_{13})(1+g_2-(1-\alpha)\omega_{23})\cdots(1+g_t-(1-\alpha)\omega_{t3})}{(1+k+(1-\alpha)\omega_6)^n}, \\ \tilde{G}_U(\alpha) &= \sum_{t=1}^{n} \frac{(D_0+(1-\alpha)\omega_2)(1+g_1+(1-\alpha)\omega_{14})(1+g_2+(1-\alpha)\omega_{24})\cdots(1+g_t+(1-\alpha)\omega_{t4})}{(1+k-(1-\alpha)\omega_5)^n} \end{aligned} \tag{4.14}$$

and

$$\tilde{H}_L(\alpha) = \frac{P_n-(1-\alpha)\omega_7}{(1+k+(1-\alpha)\omega_6)^n}, \quad \tilde{H}_U(\alpha) = \frac{P_n+(1-\alpha)\omega_8}{(1+k-(1-\alpha)\omega_5)^n}. \tag{4.15}$$

By Definition 2.4 and (4.15), for each $\lambda \in (0,1)$, using λ-signed distance method to defuzzify $\tilde{H}$, we have the following three situations of H_λ^*:

for $n\geq 3$,

$$\begin{aligned}H_\lambda^* &\equiv d(\tilde{H},\tilde{0};\lambda)\\&=\int_0^1\left[\frac{\lambda(P_n-(1-\alpha)\omega_7)}{(1+k+(1-\alpha)\omega_6)^n}+\frac{(1-\lambda)(P_n+(1-\alpha)\omega_8)}{(1+k-(1-\alpha)\omega_5)^n}\right]d\alpha\\&=\frac{\lambda}{\omega_6}\left(P_n+(1+k)\frac{\omega_7}{\omega_6}\right)\left(\frac{1}{-n+1}\right)[(1+k+\omega_6)^{-n+1}\\&\quad-(1+k)^{-n+1}]+\frac{\lambda\omega_7}{\omega_6^2}\left(\frac{1}{-n+2}\right)[(1+k)^{-n+2}\\&\quad-(1+k+\omega_6)^{-n+2}]+\frac{1-\lambda}{\omega_5}\left(P_n+(1+k)\frac{\omega_8}{\omega_5}\right)\\&\quad\times\left(\frac{1}{-n+1}\right)[(1+k)^{-n+1}-(1+k-\omega_5)^{-n+1}]\\&\quad-\frac{(1-\lambda)\omega_8}{\omega_5^2}\left(\frac{1}{-n+2}\right)[(1+k)^{-n+2}\\&\quad-(1+k-\omega_5)^{-n+2}];\end{aligned}\tag{4.16}$$

for $n=2$,

$$\begin{aligned}H_\lambda^* &\equiv d(\tilde{H},\tilde{0};\lambda)\\&=\int_0^1\left[\frac{\lambda(P_2-(1-\alpha)\omega_7)}{(1+k+(1-\alpha)\omega_6)^2}+\frac{(1-\lambda)(P_2+(1-\alpha)\omega_8)}{(1+k-(1-\alpha)\omega_5)^2}\right]d\alpha\\&=\frac{-\lambda}{\omega_6}\left(P_2+(1+k)\frac{\omega_7}{\omega_6}\right)[(1+k+\omega_6)^{-1}-(1+k)^{-1}]\\&\quad-\frac{\lambda\omega_7}{\omega_6}\ln\left(1+\frac{\omega_6}{1+k}\right)-\frac{1-\lambda}{\omega_5}\left(P_2+(1+k)\frac{\omega_8}{\omega_5}\right)[(1+k)^{-1}\\&\quad-(1+k-\omega_5)^{-1}]+\frac{(1-\lambda)\omega_8}{\omega_5^2}\ln\left(1-\frac{\omega_5}{1+k}\right);\end{aligned}\tag{4.17}$$

for $n=1$,

$$\begin{aligned}H_\lambda^* &\equiv d(\tilde{H},\tilde{0};\lambda)\\&=\frac{\lambda}{\omega_6}\left(P_1+(1+k)\frac{\omega_7}{\omega_6}\right)\ln\left(1+\frac{\omega_6}{1+k}\right)-\frac{\lambda\omega_7}{\omega_6}\\&\quad-\frac{1-\lambda}{\omega_5}\left(P_1+(1+k)\frac{\omega_8}{\omega_5}\right)\ln\left(1-\frac{\omega_5}{1+k}\right)-\frac{(1-\lambda)\omega_8}{\omega_5}.\end{aligned}\tag{4.18}$$

For each $\lambda\in(0,1)$, in order to obtain the general result of defuzzification of $\tilde{G}$ by using λ-signed distance method, we let

$t\in\{1,2,\ldots,n\},\ \theta=1-\alpha,$

and

$$\begin{aligned}&F_t(a_1,a_2,\ldots,a_{t+1},b_1,b_2,\ldots b_{t+1},c,e)\\&=\int_0^1\frac{(a_1(1-\alpha)+b_1)(a_2(1-\alpha)+b_2)\cdots(a_{t+1}(1-\alpha)+b_{t+1})}{(c(1-\alpha)+e)^t}d\alpha\\&=\int_0^1\frac{(a_1\theta+b_1)(a_2\theta+b_2)\cdots(a_{t+1}\theta+b_{t+1})}{(c\theta+e)^t}d\theta.\end{aligned}\tag{4.19}$$

From (4.19), we conduct the coefficient of θ^r of $(a_1\theta+b_1)(a_2\theta+b_2)\cdots(a_{t+1}\theta+b_{t+1})$ as below:

Let $r\in\{1,2,\ldots,t+1\}$, and A_r is a set of $(i_1,i_2,\ldots,i_r,\ i_{r+1},i_{r+2},\ldots,i_{t+1})$ satisfied the following conditions (1°), (2°).

(1°) $1\leq i_1<i_2<\cdots<i_r\leq t+1$ and $1\leq i_{r+1}<i_{r+2}<\cdots<i_{t+1}\leq t+1$.

(2°) there are no same value in $i_1,i_2,\ldots,i_r,\ i_{r+1},i_{r+2},\ldots,i_{t+1}$, and they are combined as $\{1,2,\ldots,t+1\}$.

Thus, A_r has

$$\binom{t+1}{r}=\frac{(t+1)!}{r!(t+1-r)!}$$

elements in all, and $\sum_r$ represents the sum of $a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}$ in $(i_1,i_2,\ldots,i_r,\ i_{r+1},i_{r+2},\ldots,i_{t+1})\in A_r$ then the coefficient of θ^r can be denoted as

$\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}$.

Again, we let $z=c\theta+e$, then the numerator of integration in (4.19) is

$$\begin{aligned}&(a_1\theta+b_1)(a_2\theta+b_2)\cdots(a_{t+1}\theta+b_{t+1})\\&=b_1b_2\cdots b_{t+1}+\textstyle\sum_1 a_{i_1}b_{i_2}b_{i_3}\cdots b_{i_{t+1}}\theta+\cdots\\&\quad+\textstyle\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}\theta^r+\cdots\\&\quad+\textstyle\sum_{t-1} a_{i_1}a_{i_2}\cdots a_{i_{t-1}}b_{i_t}b_{i_{t+1}}\theta^{t-1}\\&\quad+\textstyle\sum_t a_{i_1}a_{i_2}\cdots a_{i_t}b_{i_{t+1}}\theta^t+a_1a_2\cdots a_{t+1}\theta^{t+1}\\&=b_1b_2\cdots b_{t+1}+\textstyle\sum_1 a_{i_1}b_{i_2}b_{i_3}\cdots b_{i_{t+1}}\left(\frac{z-e}{c}\right)+\cdots\\&\quad+\textstyle\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}\left(\frac{z-e}{c}\right)^r+\cdots\\&\quad+\textstyle\sum_{t-1} a_{i_1}a_{i_2}\cdots a_{i_{t-1}}b_{i_t}b_{i_{t+1}}\left(\frac{z-e}{c}\right)^{t-1}\\&\quad+\textstyle\sum_t a_{i_1}a_{i_2}\cdots a_{i_t}b_{i_{t+1}}\left(\frac{z-e}{c}\right)^t+a_1a_2\cdots a_{t+1}\left(\frac{z-e}{c}\right)^{t+1}\\&=\frac{1}{c^{t+1}}\Bigg[c^{t+1}b_1b_2\cdots b_{t+1}+c^t\textstyle\sum_1 a_{i_1}b_{i_2}b_{i_3}\cdots b_{i_{t+1}}(z-e)\\&\quad+\cdots+c^{t+1-r}\textstyle\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}\\&\quad\times\displaystyle\sum_{k=0}^{r}\binom{r}{k}(-e)^{r-k}z^k+\cdots+c^2\textstyle\sum_{t-1} a_{i_1}a_{i_2}\cdots a_{i_{t-1}}b_{i_t}b_{i_{t+1}}\\&\quad\times\displaystyle\sum_{k=0}^{t-1}\binom{t-1}{k}(-e)^{t-1-k}z^k+c\textstyle\sum_t a_{i_1}a_{i_2}\cdots a_{i_t}b_{i_{t+1}}\\&\quad\times\displaystyle\sum_{k=0}^{t}\binom{t}{k}(-e)^{t-k}z^k+a_1a_2\cdots a_{t+1}\\&\quad\times\displaystyle\sum_{k=0}^{t+1}\binom{t+1}{k}(-e)^{t+1-k}z^k\Bigg]\equiv\frac{1}{c^{t+1}}h(z).\end{aligned}\tag{4.20}$$

By (4.19), (4.20) and $z=c\theta+e$, we obtain when $t\in\{1,2,\ldots,n\}$,

$$F_t(a_1,a_2,\ldots,a_{t+1},b_1,b_2,\ldots b_{t+1},c,e)$$
$$\equiv\frac{1}{c^{t+2}}\int_e^{c+e}\frac{1}{z^t}h(z)\mathrm{d}z$$
$$=\frac{1}{c^{t+2}}\int_e^{c+e}\Bigg[c^{t+1}b_1b_2\cdots b_{t+1}z^{-t}$$
$$+c^t\textstyle\sum_1 a_{i_1}b_{i_2}b_{i_3}\cdots b_{i_{t+1}}(z^{-t+1}-ez^{-t})+\cdots$$
$$+c^{t+1-r}\textstyle\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}$$
$$\times\sum_{k=0}^{r}\binom{r}{k}(-e)^{r-k}z^{-t+k}+\cdots$$
$$+c^2\textstyle\sum_{t-1} a_{i_1}a_{i_2}\cdots a_{i_{t-1}}b_{i_t}b_{i_{t+1}}\displaystyle\sum_{k=0}^{t-1}\binom{t-1}{k}(-e)^{t-1-k}z^{-t+k}$$
$$+c\textstyle\sum_t a_{i_1}a_{i_2}\cdots a_{i_t}b_{i_{t+1}}\displaystyle\sum_{k=0}^{t}\binom{t}{k}(-e)^{t-k}z^{-t+k}$$
$$+a_1a_2\cdots a_{t+1}\sum_{k=0}^{t+1}\binom{t+1}{k}(-e)^{t+1-k}z^{-t+k}\Bigg]\mathrm{d}z$$
$$=\frac{1}{c^{t+2}}\Bigg[\frac{c^{t+1}}{-t+1}b_1b_2\cdots b_{t+1}((c+e)^{-t+1}-e^{-t+1})$$
$$+c^t\textstyle\sum_1 a_{i_1}b_{i_2}b_{i_3}\cdots b_{i_{t+1}}\left[\frac{1}{-t+2}((c+e)^{-t+2}-e^{-t+2})\right.$$
$$\left.-\frac{e}{-t+1}((c+e)^{-t+1}-e^{-t+1})\right]+\cdots$$
$$+c^{t+1-r}\textstyle\sum_r a_{i_1}a_{i_2}\cdots a_{i_r}b_{i_{r+1}}b_{i_{r+2}}\cdots b_{i_{t+1}}$$
$$\times\sum_{k=0}^{r}\binom{r}{k}(-e)^{r-k}\left(\frac{1}{-t+k+1}\right)((c+e)^{-t+k+1}$$
$$-e^{-t+k+1})+\cdots+c^2\textstyle\sum_{t-1}a_{i_1}a_{i_2}\cdots a_{i_{t-1}}b_{i_t}b_{i_{t+1}}$$
$$\times\left[\sum_{k=0}^{t-2}\binom{t-1}{k}(-e)^{t-1-k}\left(\frac{1}{-t+k+1}\right)\right.$$
$$\left.\times((c+e)^{-t+k+1}-e^{-t+k+1})+\ln\left(1+\frac{c}{e}\right)\right]$$
$$+c\textstyle\sum_t a_{i_1}a_{i_2}\cdots a_{i_t}b_{i_{t+1}}\displaystyle\left[\sum_{k=0}^{t-2}\binom{t}{k}(-e)^{t-k}\left(\frac{1}{-t+k+1}\right)\right.$$
$$\left.\times((c+e)^{-t+k+1}-e^{-t+k+1})-te\ln\left(1+\frac{c}{e}\right)+e\right]$$
$$+a_{i_1}a_{i_2}\cdots a_{i_t}\left[\sum_{k=0}^{t-2}\binom{t+1}{k}(-e)^{t+1-k}\left(\frac{1}{-t+k+1}\right)\right.$$
$$\times((c+e)^{-t+k+1}-e^{-t+k+1})+\binom{t+1}{t-1}e^2\ln\left(1+\frac{c}{e}\right)$$
$$+\sum_{k=t}^{t+1}\binom{t+1}{k}(-e)^{t+1-k}\left(\frac{1}{-t+k+1}\right)$$
$$\left.\times((c+e)^{-t+k+1}-e^{-t+k+1})\right]\Bigg].\tag{4.21}$$

Then, by (4.14), (4.19), and (4.21), for each $\lambda\in(0,1)$, using λ-signed distance method to defuzzify $\tilde{G}$, then we obtain

$$G_\lambda^*\equiv\mathrm{d}(\tilde{G},\tilde{0};\lambda)=\int_0^1[\lambda\tilde{G}_{\mathrm{L}}(\alpha)+(1-\lambda)\tilde{G}_{\mathrm{U}}(\alpha)]\mathrm{d}\alpha$$
$$=\sum_{t=1}^{n}[\lambda F_t(-\omega_1,-\omega_{13},-\omega_{23},\ldots,-\omega_{t3},D_0,1+g_1,$$
$$1+g_2,\ldots,1+g_t,\omega_6,1+k)$$
$$+(1-\lambda)F_t(\omega_2,\omega_{14},\omega_{24},\ldots,\omega_{t4},D_0,1+g_1,$$
$$1+g_2,\ldots,1+g_t,-\omega_5,1+k)].\tag{4.22}$$

Therefore, by (4.1) to (4.6), (4.11), (4.12), (4.16) to (4.18), (4.22) and Property 2.9, we can obtain the following Theorem 4.1.

Theorem 4.1. *Utilizing (4.3) to (4.6) to fuzzify (4.1), we have*

(1°) fuzzy intrinsic value $\tilde{V}_0^*=\tilde{G}(+)\tilde{H}$ where $\tilde{G}$ in (4.11), $\tilde{H}$ in (4.12);
(2°) the estimation of intrinsic value in the fuzzy sense is

$$\hat{V}_{0\lambda}^*\equiv\mathrm{d}(\tilde{V}_0^*,\tilde{0};\lambda)=\mathrm{d}(\tilde{G},\tilde{0};\lambda)+\mathrm{d}(\tilde{H},\tilde{0};\lambda)=H_\lambda^*+G_\lambda^*,$$

where H_λ^* in (4.16) to (4.18), G_λ^* in (4.22).

Remark 4.2. The relation between Theorem 4.1 and crisp case is discussed in Section 6.1.

5. Numerical examples

In this section, we will illustrate the methodology given in the preceding sections to evaluate the intrinsic value with the following example under different λ levels ($\lambda=0.5$, 0.2, 0.8).

In the event that we would like to hold an asset for n years, and then sell it at a expected price P_n, so we can employ Theorem 4.1 to compute the intrinsic value in the fuzzy sense. Let $n=1,2,3$, $D_0=\$2$ (measuring unit in US Dollar), $k=6\%$, $g=3\%$, $P_n=\$35$ (measuring unit in US Dollar). In addition, we can appropriately determine the values of ω_1, ω_2, ω_{t3}, ω_{t4}, ω_5, ω_6, ω_7 and ω_8. Furthermore, in order to find the relative errors between fuzzy case ($\hat{V}_{0\lambda}^*$) and crisp case (V_0^*), we let

$$r=\frac{\hat{V}_{0\lambda}^*-V_0^*}{V_0^*}\times100\%,$$

and then we show the numerical results in Tables 1–3.

Table 1
The numerical comparisons of crisp case and fuzzy case for $n=3$ with different λ levels

Crisp	D_0	k	g_1	g_2	g_3	P_n	V_0^*													
case	2	0.06	0.03	0.03	0.03	35	35.053													
Fuzzy	$D_{0\lambda}^*$	k_λ^*	$g_{1\lambda}^*$	$g_{2\lambda}^*$	$g_{3\lambda}^*$	$P_{n\lambda}^*$	ω_1	ω_2	ω_{13}	ω_{23}	ω_{33}	ω_{14}	ω_{24}	ω_{34}	ω_5	ω_6	ω_7	ω_8	$\hat{V}_{0\lambda}^*$	$r(\%)$
case	2	0.06	0.03	0.03	0.03	35	0	0	0	0	0	0	0	0	0	0	0	0	35.053	0.153
for $\lambda=$	2	0.06	0.03	0.03	0.03	35	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.01	35.054	0.153
0.5	1.9775	0.06	0.03	0.03	0.03	35	0.1	0.01	0	0	0	0	0	0	0	0	0.01	0.01	34.926	−0.211
	2.0225	0.06	0.03	0.03	0.03	35	0.01	0.1	0	0	0	0	0	0	0	0	0.01	0.01	35.181	0.518
	2	0.06	0.0278	0.03	0.03	35	0.01	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	35.029	0.082
	2	0.06	0.03	0.0278	0.03	35	0.01	0.01	0	0.01	0	0	0	0	0	0	0.01	0.01	35.037	0.107
	2	0.06	0.03	0.03	0.0278	35	0.01	0.01	0	0	0.01	0	0	0	0	0	0.01	0.01	35.046	0.13
	2	0.06	0.0323	0.03	0.03	35	0.01	0.01	0	0	0	0.01	0	0	0	0	0.01	0.01	35.078	0.224
	2	0.06	0.03	0.0323	0.03	35	0.01	0.01	0	0	0	0	0.01	0	0	0	0.01	0.01	35.07	0.2
	2	0.06	0.03	0.03	0.0323	35	0.01	0.01	0	0	0	0	0	0.01	0	0	0.01	0.01	35.062	0.176
	2	0.0578	0.03	0.03	0.03	35	0.01	0.01	0	0	0	0	0	0	0.01	0	0.01	0.01	35.481	1.375
	2	0.0623	0.03	0.03	0.03	35	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	0.01	34.637	−1.037
	2	0.06	0.03	0.03	0.03	34.9775	0.01	0.01	0	0	0	0	0	0	0	0	0.1	0.01	35.016	0.045
	2	0.06	0.03	0.03	0.03	35.0225	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.1	35.091	0.261
	1.9775	0.0623	0.0278	0.0278	0.0278	34.9775	0.1	0.01	0.01	0.01	0.01	0	0	0	0	0.01	0.1	0.01	34.427	−1.636
	1.8775	0.0648	0.0253	0.0253	0.0253	34.8775	0.5	0.01	0.02	0.02	0.02	0	0	0	0	0.02	0.5	0.01	33.226	−5.068
	2.0225	0.0553	0.0348	0.0348	0.0348	35.1225	0.01	0.1	0	0	0	0.02	0.02	0.02	0.02	0	0.01	0.5	36.424	4.069
Fuzzy	2.003	0.0603	0.0303	0.0303	0.0303	35.003	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.01	35.082	0.235
case	1.994	0.0603	0.0303	0.0303	0.0303	35.003	0.1	0.01	0	0	0	0	0	0	0	0	0.01	0.01	35.031	0.089
for $\lambda=$	2.039	0.0603	0.0303	0.0303	0.0303	35.003	0.01	0.1	0	0	0	0	0	0	0	0	0.01	0.01	35.286	0.818
0.2	2.003	0.0603	0.0294	0.0303	0.0303	35.003	0.01	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	35.072	0.206
	2.003	0.0603	0.0303	0.0294	0.0303	35.003	0.01	0.01	0	0.01	0	0	0	0	0	0	0.01	0.01	35.076	0.216
	2.003	0.0603	0.0303	0.0303	0.0294	35.003	0.01	0.01	0	0	0.01	0	0	0	0	0	0.01	0.01	35.079	0.225
	2.003	0.0603	0.0339	0.0303	0.0303	35.003	0.01	0.01	0	0	0	0.01	0	0	0	0	0.01	0.01	35.122	0.348
	2.003	0.0603	0.0303	0.0339	0.0303	35.003	0.01	0.01	0	0	0	0	0.01	0	0	0	0.01	0.01	35.108	0.309
	2.003	0.0603	0.0303	0.0303	0.0339	35.003	0.01	0.01	0	0	0	0	0	0.01	0	0	0.01	0.01	35.095	0.271
	2.003	0.0594	0.0303	0.0303	0.0303	35.003	0.01	0.01	0	0	0	0	0	0	0.01	0	0.01	0.01	35.51	1.457
	2.003	0.0639	0.0303	0.0303	0.0303	35.003	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	0.01	34.665	−0.958
	2.003	0.0603	0.0303	0.0303	0.0303	34.994	0.01	0.01	0	0	0	0	0	0	0	0	0.1	0.01	35.067	0.191
	2.003	0.0603	0.0303	0.0303	0.0303	35.039	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.1	35.143	0.407
	1.994	0.0639	0.0294	0.0294	0.0294	34.994	0.1	0.01	0.01	0.01	0.01	0	0	0	0	0.01	0.1	0.01	34.58	−1.199
	1.954	0.0679	0.0284	0.0284	0.0284	34.954	0.5	0.01	0.02	0.02	0.02	0	0	0	0	0.02	0.5	0.01	33.825	−3.357
	2.039	0.0584	0.0379	0.0379	0.0379	35.199	0.01	0.1	0	0	0	0.02	0.02	0.02	0.02	0	0.01	0.5	36.727	4.934
Fuzzy	1.997	0.0597	0.0297	0.0297	0.0297	34.997	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.01	35.025	0.071
case	1.961	0.0597	0.0297	0.0297	0.0297	34.997	0.1	0.01	0	0	0	0	0	0	0	0	0.01	0.01	34.821	−0.511
for $\lambda=$	2.006	0.0597	0.0297	0.0297	0.0297	34.997	0.01	0.1	0	0	0	0	0	0	0	0	0.01	0.01	35.076	0.217
0.8	1.997	0.0597	0.0261	0.0297	0.0297	34.997	0.01	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	34.986	−0.041
	1.997	0.0597	0.0297	0.0261	0.0297	34.997	0.01	0.01	0	0.01	0	0	0	0	0	0	0.01	0.01	34.999	−0.003
	1.997	0.0597	0.0297	0.0297	0.0261	34.997	0.01	0.01	0	0	0.01	0	0	0	0	0	0.01	0.01	35.012	0.035
	1.997	0.0597	0.0306	0.0297	0.0297	34.997	0.01	0.01	0	0	0	0.01	0	0	0	0	0.01	0.01	35.035	0.1
	1.997	0.0597	0.0297	0.0306	0.0297	34.997	0.01	0.01	0	0	0	0	0.01	0	0	0	0.01	0.01	35.031	0.09
	1.997	0.0597	0.0297	0.0297	0.0306	34.997	0.01	0.01	0	0	0	0	0	0.01	0	0	0.01	0.01	35.028	0.08
	1.997	0.0561	0.0297	0.0297	0.0297	34.997	0.01	0.01	0	0	0	0	0	0	0.01	0	0.01	0.01	35.451	1.29

(continued on next page)

Table 1 (*continued*)

1.997	0.0606	0.0297	0.0297	0.0297	34.997	0.01	0.01	0	0	0	0	0	0	0	0.01	0.01	0.01	34.609	−1.118
1.997	0.0597	0.0297	0.0297	0.0297	34.961	0.01	0.01	0	0	0	0	0	0	0	0	0.1	0.01	34.965	−0.101
1.997	0.0597	0.0297	0.0297	0.0297	35.006	0.01	0.01	0	0	0	0	0	0	0	0	0.01	0.1	35.04	0.114
1.961	0.0606	0.0261	0.0261	0.0261	34.961	0.1	0.01	0.01	0.01	0.01	0	0	0	0	0.01	0.1	0.01	34.274	−2.074
1.801	0.0616	0.0221	0.0221	0.0221	34.801	0.5	0.01	0.02	0.02	0.02	0	0	0	0	0.02	0.5	0.01	32.628	−6.777
2.006	0.0521	0.0316	0.0316	0.0316	35.046	0.01	0.1	0	0	0	0.02	0.02	0.02	0.02	0	0.01	0.5	36.118	3.194

From Tables 1–3, if $\omega_i \equiv \omega$, and if $\omega \to 0$, then the fuzzy case of Theorem 4.1 will approach the crisp case. It implies that if ω smaller is, then the estimated intrinsic value of fuzzy case is getting closer to crisp case.

Furthermore, from the numerical results, we find that if λ is closer to *0*, then the estimated intrinsic value $\hat{V}^*_{0\lambda}$ is greater; if λ is closer to 1, then the estimated intrinsic value $\hat{V}^*_{0\lambda}$ is smaller.

6. Discussions

Because we are interested in applying the fuzzy logic to solve the classical DCF model for valuation in which the cash flows and the discount rate are uncertainty, we have developed a FDCF model that allows us to employ triangular fuzzy numbers to explicitly analyze and provide insights into how the intrinsic value could be impacted by the vague levels of cash flow, growth rate, discount rate, and future selling price. In conclusion, our work has provided the following aspects for the proposed FDCF model.

6.1. The relation between Theorem 4.1 and crisp case (in (3.3))

(a) In (4.14), (4.15), if we assume $\omega_1 = \omega_2 = \omega_5 = \omega_6 = \omega_7 = \omega_8 = 0$ and $\omega_{j3} = \omega_{j4} = 0$ for all j, then we can obtain

$$\tilde{G}_L(\alpha) = \tilde{G}_U(\alpha) = \sum_{t=1}^{n} \frac{D_0(1+g_1)(1+g_2)\cdots(1+g_t)}{(1+k)^n};$$

$$\tilde{H}_L(\alpha) = \tilde{H}_U(\alpha) = \frac{P_n}{(1+k)^n}.$$

From Theorem 4.1 (1°),

$$\begin{aligned} d(\tilde{V}^*_0, \tilde{0}; \lambda) &= d(\tilde{G}, \tilde{0}; \lambda) + d(\tilde{H}, \tilde{0}; \lambda) \\ &= \int_0^1 [\lambda \tilde{G}_L(\alpha) + (1-\lambda)\tilde{G}_U(\alpha) + \lambda \tilde{H}_L(\alpha) \\ &\quad + (1-\lambda)\tilde{H}_U(\alpha)]\mathrm{d}\alpha \\ &= V^*_0 \ \ (\text{in}(3.3)). \end{aligned}$$

(b) In crisp case, if the investor only considers to hold the specific asset until the *n*th year, by assuming $P_n = 0$, then we can obtain

$$V^*_0 = \sum_{t=1}^{n} \frac{D_t}{(1+k)^t},$$

$D_t = D_{t-1}(1+g_t)$, $t = 1,2,\ldots,n$.

(c) Considering the fuzzy case in Theorem 4.1, if $P_n = 0$, $\omega_7 = \omega_8 = 0$, then $\tilde{P}_n = \tilde{0}$ (i.e. a fuzzy point at 0). By (4.11) and (4.12), we can obtain $\tilde{G}$ (in (4.11)) and $\tilde{H} = \tilde{0}$ (in (4.12)). Thus, from Theorem 4.1 (1°), the fuzzy intrinsic value is $\tilde{V}^*_0 = \tilde{G}(+)\tilde{0}$. That is the estimates intrinsic value $\hat{V}^*_{0\lambda} \equiv d(\tilde{V}^*_0, \tilde{0}; \lambda) = d(\tilde{G}, \tilde{0}; \lambda) = G^*_\lambda$ (in Theorem 4.1 (2°)).

Table 2
The numerical comparisons of crisp case and fuzzy case for $n=2$ with different λ levels

Crisp	D_0	k	g_1	g_2	g_3	P_n	V_0^*													
case	2	0.06	0.03	0.03	−	35	34.982													
Fuzzy	$D_{0\lambda}^*$	k_λ^*	$g_{1\lambda}^*$	$g_{2\lambda}^*$	$g_{3\lambda}^*$	$P_{n\lambda}^*$	ω_1	ω_2	ω_{13}	ω_{23}	ω_{33}	ω_{14}	ω_{24}	ω_{34}	ω_5	ω_6	ω_7	ω_8	$\hat{V}_{0\lambda}^*$	$r(\%)$
case	2	0.06	0.03	0.03	–	35	0	0	0	0	–	0	0	–	0	0	0	0	34.982	−0.052
for $\lambda=$	2	0.06	0.03	0.03	–	35	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.01	34.982	−0.052
0.5	1.9775	0.06	0.03	0.03	–	35	0.1	0.01	0	0	–	0	0	–	0	0	0.01	0.01	34.896	−0.298
	2.0225	0.06	0.03	0.03	–	35	0.01	0.1	0	0	–	0	0	–	0	0	0.01	0.01	35.068	0.194
	2	0.06	0.0278	0.03	–	35	0.01	0.01	0.01	0	–	0	0	–	0	0	0.01	0.01	34.965	−0.1
	2	0.06	0.03	0.0278	–	35	0.01	0.01	0	0.01	–	0	0	–	0	0	0.01	0.01	34.973	−0.076
	2	0.06	0.0323	0.03	–	35	0.01	0.01	0	0	–	0.01	0	–	0	0	0.01	0.01	34.998	−0.004
	2	0.06	0.03	0.0323	–	35	0.01	0.01	0	0	–	0	0.01	–	0	0	0.01	0.01	34.99	−0.029
	2	0.0578	0.03	0.03	–	35	0.01	0.01	0	0	–	0	0	–	0.01	0	0.01	0.01	35.273	0.781
	2	0.0623	0.03	0.03	–	35	0.01	0.01	0	0	–	0	0	–	0	0.01	0.01	0.01	34.696	−0.869
	2	0.06	0.03	0.03	–	34.9775	0.01	0.01	0	0	–	0	0	–	0	0	0.1	0.01	34.942	−0.167
	2	0.06	0.03	0.03	–	35.0225	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.1	35.022	0.062
	1.9775	0.0623	0.0278	0.0278	–	34.9775	0.1	0.01	0.01	0.01	–	0	0	–	0	0.01	0.1	0.01	34.547	−1.295
	1.8775	0.0648	0.0253	0.0253	–	34.8775	0.5	0.01	0.02	0.02	–	0	0	–	0	0.02	0.5	0.01	33.666	−3.812
	2.0225	0.0553	0.0348	0.0348	–	35.1225	0.01	0.1	0	0	–	0.02	0.02	−	0.02	0	0.01	0.5	35.971	2.774
Fuzzy	2.003	0.0603	0.0303	0.0303	–	35.003	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.01	35.002	0.005
case	1.994	0.0603	0.0303	0.0303	–	35.003	0.1	0.01	0	0	–	0	0	–	0	0	0.01	0.01	34.967	−0.093
for $\lambda=$	2.039	0.0603	0.0303	0.0303	–	35.003	0.01	0.1	0	0	–	0	0	–	0	0	0.01	0.01	35.14	0.4
0.2	2.003	0.0603	0.0294	0.0303	–	35.003	0.01	0.01	0.01	0	–	0	0	–	0	0	0.01	0.01	34.995	−0.014
	2.003	0.0603	0.0303	0.0294	–	35.003	0.01	0.01	0	0.01	–	0	0	–	0	0	0.01	0.01	34.999	−0.004
	2.003	0.0603	0.0339	0.0303	–	35.003	0.01	0.01	0	0	–	0.01	0	–	0	0	0.01	0.01	35.029	0.082
	2.003	0.0603	0.0303	0.0339	–	35.003	0.01	0.01	0	0	–	0	0.01	–	0	0	0.01	0.01	35.015	0.043
	2.003	0.0594	0.0303	0.0303	–	35.003	0.01	0.01	0	0	–	0	0	–	0.01	0	0.01	0.01	35.294	0.839
	2.003	0.0639	0.0303	0.0303	–	35.003	0.01	0.01	0	0	–	0	0	–	0	0.01	0.01	0.01	34.715	−0.813
	2.003	0.0603	0.0303	0.0303	–	34.994	0.01	0.01	0	0	–	0	0	–	0	0	0.1	0.01	34.986	−0.04
	2.003	0.0603	0.0303	0.0303	–	35.039	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.1	35.066	0.189
	1.994	0.0639	0.0294	0.0294	–	34.994	0.1	0.01	0.01	0.01	–	0	0	–	0	0.01	0.1	0.01	34.656	−0.984
	1.954	0.0679	0.0284	0.0284	–	34.954	0.5	0.01	0.02	0.02	–	0	0	–	0	0.02	0.5	0.01	34.114	−2.531
	2.039	0.0584	0.0379	0.0379	–	35.199	0.01	0.1	0	0	–	0.02	0.02	–	0.02	0	0.01	0.5	36.211	3.461
Fuzzy	1.997	0.0597	0.0297	0.0297	–	34.997	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.01	34.962	−0.11
case	1.961	0.0597	0.0297	0.0297	–	34.997	0.1	0.01	0	0	–	0	0	–	0	0	0.01	0.01	34.824	−0.504
for $\lambda=$	2.006	0.0597	0.0297	0.0297	–	34.997	0.01	0.1	0	0	–	0	0	–	0	0	0.01	0.01	34.996	−0.011
0.8	1.997	0.0597	0.0261	0.0297	–	34.997	0.01	0.01	0.01	0	–	0	0	–	0	0	0.01	0.01	34.935	−0.186
	1.997	0.0597	0.0297	0.0261	–	34.997	0.01	0.01	0	0.01	–	0	0	–	0	0	0.01	0.01	34.948	−0.147
	1.997	0.0597	0.0306	0.0297	–	34.997	0.01	0.01	0	0	–	0.01	0	–	0	0	0.01	0.01	34.968	−0.091
	1.997	0.0597	0.0297	0.0306	–	34.997	0.01	0.01	0	0	–	0	0.01	–	0	0	0.01	0.01	34.965	−0.1
	1.997	0.0561	0.0297	0.0297	–	34.997	0.01	0.01	0	0	–	0	0	–	0.01	0	0.01	0.01	35.253	0.722
	1.997	0.0606	0.0297	0.0297	–	34.997	0.01	0.01	0	0	–	0	0	–	0	0.01	0.01	0.01	34.676	−0.926
	1.997	0.0597	0.0297	0.0297	–	34.961	0.01	0.01	0	0	–	0	0	–	0	0	0.1	0.01	34.897	−0.293
	1.997	0.0597	0.0297	0.0297	–	35.006	0.01	0.01	0	0	–	0	0	–	0	0	0.01	0.1	34.978	−0.064
	1.961	0.0606	0.0261	0.0261	–	34.961	0.1	0.01	0.01	0.01	–	0	0	–	0	0.01	0.1	0.01	34.438	−1.607
	1.801	0.0616	0.0221	0.0221	–	34.801	0.5	0.01	0.02	0.02	–	0	0	–	0	0.02	0.5	0.01	33.218	−5.092
	2.006	0.0521	0.0316	0.0316	–	35.046	0.01	0.1	0	0	–	0.02	0.02	–	0.02	0	0.01	0.5	35.728	2.081

Table 3
The numerical comparisons of crisp case and fuzzy case for $n=1$ with different λ levels

Crisp	D_0	k	g_1	g_2	g_3	P_n	V_0^*													
case	2	0.06	0.03	–	–	35	34.962													
Fuzzy	$D_{0\lambda}^*$	k_λ^*	$g_{1\lambda}^*$	$g_{2\lambda}^*$	$g_{3\lambda}^*$	$P_{n\lambda}^*$	ω_1	ω_2	ω_{13}	ω_{23}	ω_{33}	ω_{14}	ω_{24}	ω_{34}	ω_5	ω_6	ω_7	ω_8	$\hat{V}_{0\lambda}^*$	$r(\%)$
case	2	0.06	0.03	–	–	35	0	0	0	–	–	0	–	–	0	0	0	0	34.962	−0.108
for $\lambda=$	2	0.06	0.03	–	–	35	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.962	−0.108
0.5	1.9775	0.06	0.03	–	–	35	0.1	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.919	−0.233
	2.0225	0.06	0.03	–	–	35	0.01	0.1	0	–	–	0	–	–	0	0	0.01	0.01	35.006	0.017
	2	0.06	0.0278	–	–	35	0.01	0.01	0.01	–	–	0	–	–	0	0	0.01	0.01	34.954	−0.132
	2	0.06	0.0323	–	–	35	0.01	0.01	0	–	–	0.01	–	–	0	0	0.01	0.01	34.971	−0.083
	2	0.0578	0.03	–	–	35	0.01	0.01	0	–	–	0	–	–	0.01	0	0.01	0.01	35.112	0.319
	2	0.0623	0.03	–	–	35	0.01	0.01	0	–	–	0	–	–	0	0.01	0.01	0.01	34.815	−0.529
	2	0.06	0.03	–	–	34.9775	0.01	0.01	0	–	–	0	–	–	0	0	0.1	0.01	34.92	−0.229
	2	0.06	0.03	–	–	35.0225	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.1	35.005	0.014
	1.9775	0.0623	0.0278	–	–	34.9775	0.1	0.01	0.01	–	–	0	–	–	0	0.01	0.1	0.01	34.721	−0.797
	1.8775	0.0648	0.0253	–	–	34.8775	0.5	0.01	0.02	–	–	0	–	–	0	0.02	0.5	0.01	34.175	−2.356
	2.0225	0.0553	0.0348	–	–	35.1225	0.01	0.1	0	–	–	0.02	–	–	0.02	0	0.01	0.5	35.577	1.649
Fuzzy	2.003	0.0603	0.0303	–	–	35.003	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.975	−0.072
case	1.994	0.0603	0.0303	–	–	35.003	0.1	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.957	−0.122
for $\lambda=$	2.039	0.0603	0.0303	–	–	35.003	0.01	0.1	0	–	–	0	–	–	0	0	0.01	0.01	35.045	0.128
0.2	2.003	0.0603	0.0294	–	–	35.003	0.01	0.01	0.01	–	–	0	–	–	0	0	0.01	0.01	34.972	−0.081
	2.003	0.0603	0.0339	–	–	35.003	0.01	0.01	0	–	–	0.01	–	–	0	0	0.01	0.01	34.989	−0.033
	2.003	0.0594	0.0303	–	–	35.003	0.01	0.01	0	–	–	0	–	–	0.01	0	0.01	0.01	35.124	0.355
	2.003	0.0639	0.0303	–	–	35.003	0.01	0.01	0	–	–	0	–	–	0	0.01	0.01	0.01	34.827	−0.494
	2.003	0.0603	0.0303	–	–	34.994	0.01	0.01	0	–	–	0	–	–	0	0	0.1	0.01	34.958	−0.12
	2.003	0.0603	0.0303	–	–	35.039	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.1	35.043	0.122
	1.994	0.0639	0.0294	–	–	34.994	0.1	0.01	0.01	–	–	0	–	–	0	0.01	0.1	0.01	34.79	−0.601
	1.954	0.0679	0.0284	–	–	34.954	0.5	0.01	0.02	–	–	0	–	–	0	0.02	0.5	0.01	34.474	−1.504
	2.039	0.0584	0.0379	–	–	35.199	0.01	0.1	0	–	–	0.02	–	–	0.02	0	0.01	0.5	35.768	2.194
Fuzzy	1.997	0.0597	0.0297	–	–	34.997	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.95	−0.144
case	1.961	0.0597	0.0297	–	–	34.997	0.1	0.01	0	–	–	0	–	–	0	0	0.01	0.01	34.88	−0.344
for $\lambda=$	2.006	0.0597	0.0297	–	–	34.997	0.01	0.1	0	–	–	0	–	–	0	0	0.01	0.01	34.967	−0.094
0.8	1.997	0.0597	0.0261	–	–	34.997	0.01	0.01	0.01	–	–	0	–	–	0	0	0.01	0.01	34.936	−0.182
	1.997	0.0597	0.0306	–	–	34.997	0.01	0.01	0	–	–	0.01	–	–	0	0	0.01	0.01	34.953	−0.134
	1.997	0.0561	0.0297	–	–	34.997	0.01	0.01	0	–	–	0	–	–	0.01	0	0.01	0.01	35.099	0.283
	1.997	0.0606	0.0297	–	–	34.997	0.01	0.01	0	–	–	0	–	–	0	0.01	0.01	0.01	34.802	−0.565
	1.997	0.0597	0.0297	–	–	34.961	0.01	0.01	0	–	–	0	–	–	0	0	0.1	0.01	34.882	−0.338
	1.997	0.0597	0.0297	–	–	35.006	0.01	0.01	0	–	–	0	–	–	0	0	0.01	0.1	34.967	−0.095
	1.961	0.0606	0.0261	–	–	34.961	0.1	0.01	0.01	–	–	0	–	–	0	0.01	0.1	0.01	34.652	−0.993
	1.801	0.0616	0.0221	–	–	34.801	0.5	0.01	0.02	–	–	0	–	–	0	0.02	0.5	0.01	33.877	−3.209
	2.006	0.0521	0.0316	–	–	35.046	0.01	0.1	0	–	–	0.02	–	–	0.02	0	0.01	0.5	35.386	1.102

6.2. The explanation of using the λ-signed distance

In this paper, the use of λ-signed distance is based on the consideration of natural extension. The reasons for such extension are interpreted as follows.

By Yao and Wu (2000), the signed distance of a on $\Re$ measured from the origin 0 is defined by $d_0(a,0)=a$ that can be acquired by the characteristic of a real line, and such a viewpoint can be extended to that of the signed distance of fuzzy sets on F_s. Because fuzzy set $\tilde{D}(\in F_s)$ is not a real number, we must consider the signed distance from the membership function curve of $\tilde{D}$ to Y-axis (see Fig. 1) as the signed distance of $\tilde{D}$ to $\tilde{0}$, which is described as follows: By the *α-cut* method, for each $\alpha\in[0,1]$, considering the *α-cut* of $\tilde{D}$, we have *α-level* set $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$ and from Fig. 1, we can obtain the end points of line segment $\bar{PQ}$, $P(\tilde{D}_L(\alpha),\alpha)$ and $Q(\tilde{D}_U(\alpha),\alpha)$, which are on the membership function curve of $\tilde{D}$. The X-coordinates of the end points (P and Q) are $\tilde{D}_L(\alpha)$ and $\tilde{D}_U(\alpha)$, which are corresponding to the end points of *α-level* set $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$. For each $\lambda\in(0,1)$, the weighted average of $\tilde{D}_L(\alpha)$ and $\tilde{D}_U(\alpha)$ denoted by $\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$ is the inner point of *α-level* set $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$, and $R(\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha),\alpha)$ is the inner point of line segment $\bar{PQ}$ (see Fig. 1), hence the λ-signed distance from interval $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$ to origin 0 is defined as the signed distance from inner point $\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$ to origin 0. By Yao and Wu's (2000) definition, we can obtain $d_0([\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)],0;\lambda)=\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$. Hence, for each $\alpha\in[0,1]$, we have the following one-to-one mapping relations:

$[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha);\alpha]\leftrightarrow[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]\leftrightarrow\bar{PQ}$ and $\tilde{0}\leftrightarrow 0$. Thus, the λ-signed distance from $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha);\alpha]$ to $\tilde{0}$ can be defined as $d([\tilde{D}_L(\alpha),\tilde{D}_U(\alpha);\alpha],\tilde{0};\lambda)=d_0([\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)],0;\lambda)=\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$.((※))

The Eq. ((※)) is denoted as the signed distance from inner point $R(\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha),\alpha)$ of $\bar{PQ}$ to Y-axis (see Fig. 1). Because $\tilde{D}\in F_s$, Eq. ((※)) represents a continuous function with respect to α, where $0\le\alpha\le1$. In addition, since α only varies during the interval [0,1], the λ-signed distance from fuzzy set $\tilde{D}$ to $\tilde{0}$ can be found by calculating the mean value of signed distance from the inner point $(R(\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha),\alpha))$ of $\bar{PQ}$ to Y-axis. Therefore, by Definition 2.4, we have

$$d(\tilde{D},\tilde{0};\lambda)=\int_0^1[\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)]\mathrm{d}\alpha.$$

Such a function can be regarded as the λ-signed distance from fuzzy set $\tilde{D}$ to fuzzy point $\tilde{0}$. From Remark 2.12, for each $\lambda\in(0,1)$, $d(\tilde{a},\tilde{0};\lambda)=a=d_0(a,0;\lambda)$ for all $a\in\Re$, and family of all fuzzy points $\subset F_s$, thus the λ-signed distance (d) on F_s is one extension of the λ-signed distance (d_0) on $\Re$. In addition, by Properties 2.7, 2.8 and Remark 2.12, for $\lambda\in(0,1)$, fuzzy system $(F_s,d,\prec,\approx)$ is also one extension of a real system $(\Re,d_0,<,=)$.

6.3. The results of using the λ-signed distance method to defuzzify $\tilde{D}_0$ (in (4.3)), $\tilde{g}_j$ (in (4.4)), $\tilde{k}$ (in (4.5), and $\tilde{P}_n$ (in (4.6)) with different λ levels

When $\lambda<0.5$, $\lambda<0.5<(1-\lambda)$, by Fig. 1, for each $\alpha\in[0,1]$, the point $\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)$ in $[\tilde{D}_L(\alpha),\tilde{D}_U(\alpha)]$ will be closer to the right-end point $\tilde{D}_U(\alpha)$. In addition, because $0<\tilde{D}_L(\alpha)<\tilde{D}_U(\alpha)$ for all $\alpha\in[0,1]$, we have

$$\begin{aligned}&\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)\\&\quad=\tilde{D}_U(\alpha)-\lambda(\tilde{D}_U(\alpha)-\tilde{D}_L(\alpha))>\tilde{D}_U(\alpha)\\&\qquad-0.5(\tilde{D}_U(\alpha)-\tilde{D}_L(\alpha))\\&\quad=0.5\tilde{D}_L(\alpha)+0.5\tilde{D}_U(\alpha)\end{aligned}$$

for all $\alpha\in[0,1]$. Accordingly,

$$\begin{aligned}D_\lambda^*&=d(\tilde{D},\tilde{0};\lambda)\\&=\int_0^1[\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)]\mathrm{d}\alpha\\&>\int_0^1[0.5\tilde{D}_L(\alpha)+0.5\tilde{D}_U(\alpha)]\mathrm{d}\alpha\\&=d(\tilde{D},\tilde{0};0.5)=D_{0.5}^*.\end{aligned}$$

Contrarily, when $\lambda>0.5$, for each $\alpha\in[0,1]$,

$$\lambda\tilde{D}_L(\alpha)+(1-\lambda)\tilde{D}_U(\alpha)<0.5\tilde{D}_L(\alpha)+0.5\tilde{D}_U(\alpha),$$

and then $D_\lambda^*<D_{0.5}^*$. Based on the above derivations, we can obtain the same relations corresponding to $\tilde{D}_0$, $\tilde{g}_j$, $\tilde{k}$, and $\tilde{P}_n$ as follows.

When $\lambda<0.5$, then $D_{0\lambda}^*>D_{0.5}^*$, $g_{j\lambda}^*>g_{j0.5}^*$, $k_\lambda^*>k_{0.5}^*$, and $\tilde{P}_{n\lambda}>\tilde{P}_{n0.5}$; when $\lambda>0.5$, then $D_{0\lambda}^*<D_{0.5}^*$, $g_{j\lambda}^*<g_{j0.5}^*$, $k_\lambda^*<k_{0.5}^*$, and $\tilde{P}_{n\lambda}<\tilde{P}_{n0.5}$.

The above-mentioned relations may refer to the numerical results in Tables 1–3.

Therefore, in our FDCF model, it implies that the use of λ level can be regarded as a simple concept describing the investor's attitude to risk. That is, if $\lambda<0.5$, then we may denote that such an investor is an optimist when estimating the values of fuzzy numbers such as fuzzy cash flow ($\tilde{D}_0$), fuzzy growth rate ($\tilde{g}_j$), fuzzy discount rate ($\tilde{k}$), and fuzzy future selling price ($\tilde{P}_n$); if $\lambda>0.5$, then such an investor is a pessimist when estimating them. Also, if $\lambda=0.5$, then such an investor is a neutral to risk.

7. Concluding remarks

Valuation analysis is quite import to obtain a 'fair value' for an asset but also to take into account the investor's risk aversion. In this paper, we have proposed a more practical tool to deal with uncertainty and risk for a valuation model. This study extended the classical DCF model by developing a fuzzy logic system that it takes vague cash flow, growth rate, and discount rate into account in order to explicitly discuss the more realistic valuation model. In such a FDCF model, the uncertain information will be fuzzified as triangular fuzzy numbers so that it would be useful for typical investors to analyze the intrinsic value of a specific asset. We also find that the FDCF model is one extension of the classical (crisp) DCF model.

Furthermore, the success of this model is demonstrated through numerical examples that a novel fuzzy philosophy achieves a more reasonable operation on valuation, and it reveals some properties leading to a good method of helping the typical investors to master their assets' values.

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